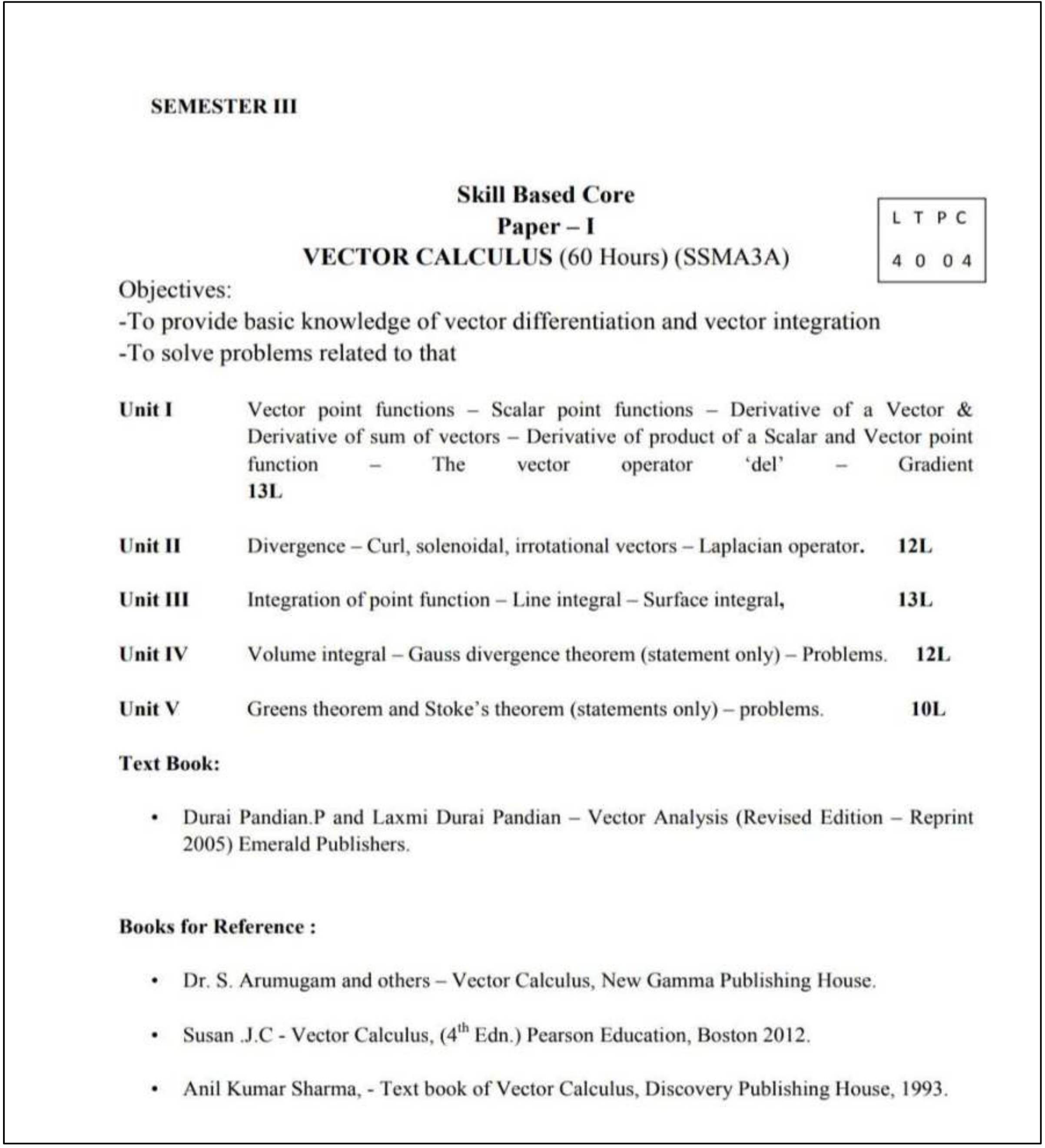
GOVERNMENT ARTS AND SCIENCE COLLEG, KOVILPATTI – 628 503.
(AFFILIATED TO MANONMANIAM SUNDARANAR UNIVERSITY, TIRUNELVELI)
DEPARTMENT OF MATHEMATICS
STUDY E - MATERIAL

CLASS : II B.SC (MATHEMATICS) SEM: III
SUBJECT : VECTOR CALCULUS (SMMA3A)



SEMESTER III

Skill Based Core
Paper – I
VECTOR CALCULUS (60 Hours) (SSMA3A)

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Objectives:

-To provide basic knowledge of vector differentiation and vector integration

-To solve problems related to that

Unit I	Vector point functions – Scalar point functions – Derivative of a Vector & Derivative of sum of vectors – Derivative of product of a Scalar and Vector point function – The vector operator 'del' – Gradient	**13L**
Unit II	Divergence – Curl, solenoidal, irrotational vectors – Laplacian operator.	**12L**
Unit III	Integration of point function – Line integral – Surface integral,	**13L**
Unit IV	Volume integral – Gauss divergence theorem (statement only) – Problems.	**12L**
Unit V	Greens theorem and Stoke's theorem (statements only) – problems.	**10L**

Text Book:

- Durai Pandian.P and Laxmi Durai Pandian – Vector Analysis (Revised Edition – Reprint 2005) Emerald Publishers.

Books for Reference :

- Dr. S. Arumugam and others – Vector Calculus, New Gamma Publishing House.
- Susan .J.C - Vector Calculus, (4^{th} Edn.) Pearson Education, Boston 2012.
- Anil Kumar Sharma, - Text book of Vector Calculus, Discovery Publishing House, 1993.

4/8/2020

Vector Calculus

Unit - I

Vector point functions - Scalar point functions - Derivative of a Vector & Derivative of sum of vectors - Derivative of product of a Scalar and Vector point function - The vector operator 'del' - Gradient

Unit - II

Divergence - Curl, solenoidal, irrotational vectors - laplacian operator.

Unit - III

Integration of point function - Line integral - Surface integral.

Unit - IV

Volume integral - Gauss divergence theorem (statement only) - Problems.

Unit - V

Greens theorem and stoke's theorem (statement only) - Problems.

Text book:

Durai Pandian. P and Laxmi Durai Pandian - Vector Analysis Revised Edition - Reprint

2005) Emerald Publishers.

Formula

1) If $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ then

$$r = |\vec{r}| = \sqrt{x^2 + y^2 + z^2}$$

2) Direction cosines of $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ are

$$\left(\frac{x}{|\vec{r}|}, \frac{y}{|\vec{r}|}, \frac{z}{|\vec{r}|} \right)$$

3) If $\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$, $\vec{b} = b_1\vec{i} + b_2\vec{j} + b_3\vec{k}$

Then $\vec{a} \cdot \vec{b} = |\vec{a}| \cdot |\vec{b}| \cos\theta$

$$\vec{a} \cdot \vec{b} = a_1b_1 + a_2b_2 + a_3b_3$$

4) $$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$\vec{a} \times \vec{b} = |\vec{a}| |\vec{b}| \sin\theta \, \hat{n}$$

5) Angle between $\vec{a}$ and $\vec{b}$

$$\cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|}$$

6) $\vec{i} \cdot \vec{i} = \vec{j} \cdot \vec{j} = \vec{k} \cdot \vec{k} = 1$

$\vec{i} \times \vec{j} = \vec{k}$ $\quad$ $\vec{j} \times \vec{i} = -\vec{k}$

$\vec{j} \times \vec{k} = \vec{i}$ $\quad$ $\vec{k} \times \vec{j} = -\vec{i}$

$\vec{k} \times \vec{i} = \vec{j}$ $\quad$ $\vec{i} \times \vec{k} = -\vec{j}$

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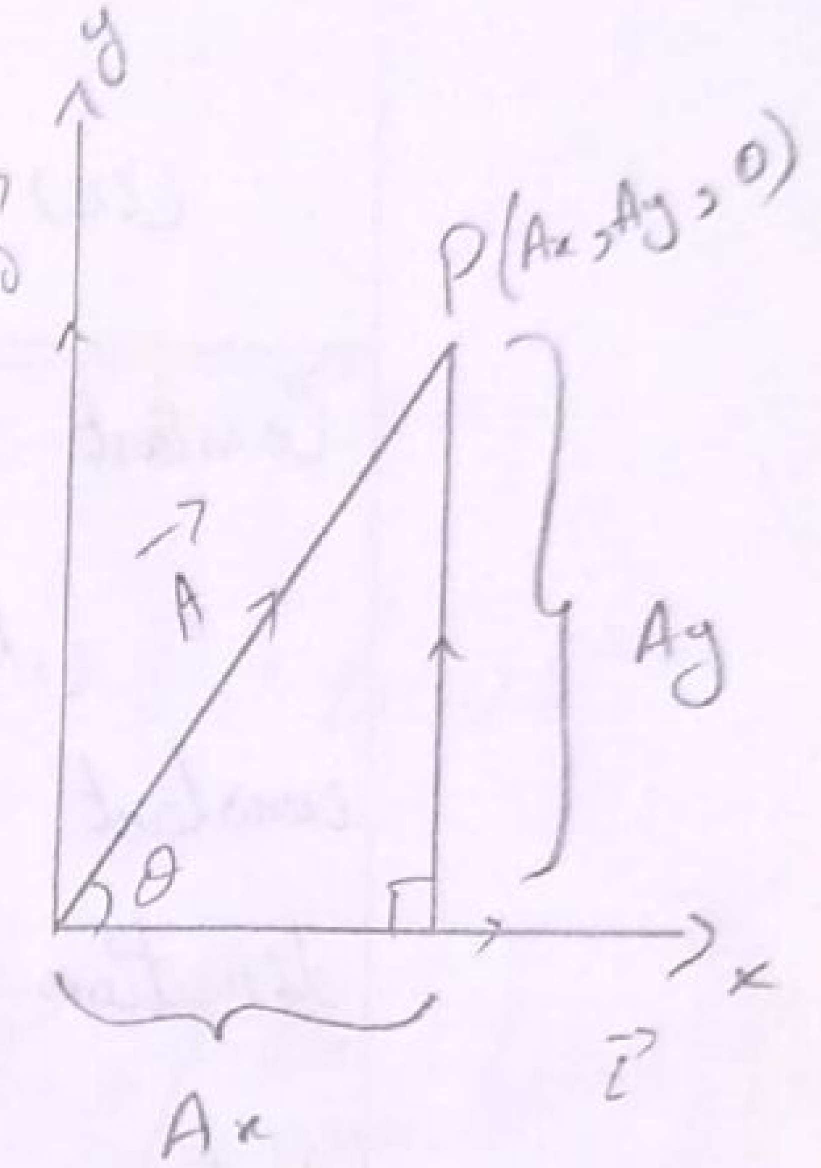
Unit - I

Notation:

$$\vec{A} = A_x \vec{i} + A_y \vec{j} + A_z \vec{k} = (A_x , A_y , A_z)$$

Remark:

Consider $\vec{A} = (A_x , A_y , 0)$



$$\cos\theta = \frac{A_x}{A} \Rightarrow A_x = A\cos\theta$$

$$\sin\theta = \frac{A_y}{A} \Rightarrow A_y = A\sin\theta$$

$$\vec{A} = A\cos\theta\,\vec{i} + A\sin\theta\,\vec{j}$$

$A_x > 0 \ , \ A_y > 0 \Rightarrow 0 < \theta < \frac{\pi}{2} \rightarrow I^{st}$ quadrant

$A_x < 0 \ , \ A_y > 0 \Rightarrow \frac{\pi}{2} < \theta < \pi \rightarrow II^{nd}$ quadrant

$A_x < 0 \ , \ A_y < 0 \Rightarrow \pi < \theta < \frac{3\pi}{2} \rightarrow III^{rd}$ quadrant

$A_x > 0 \ , \ A_y < 0 \Rightarrow -\frac{\pi}{2} < \theta < 0 \rightarrow IV^{th}$ quadrant

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Vector function

If for each value of a scalar variable u, there corresponds a vector $\vec{f}$, the $\vec{f}$ is said to a vector function of the scalar variable u. Then vector function is write as $\vec{f}(u)$.

ex: $\vec{f}(u) = (a\cos u)\vec{i} + (a\sin u)\vec{j} + bu\vec{k}$

Note:

If $\vec{f}(u)$ is a vector function of u and if $\vec{f}(u) = f_1\vec{i} + f_2\vec{j} + f_3\vec{k}$ then f_1, f_2, f_3 are functions of u.

(ie) $\vec{f}(u) = f_1(u)\vec{i} + f_2(u)\vec{j} + f_3(u)\vec{k}$

Constant Vector:

A vector whose magnitude is a constant and direction is in a fixed direction is a constant vector.

Note:

A scalar function has only a magnitude while a vector function has both magnitude and direction.

ex:

mass, Temperature - Scalars

Position, Displacement
Velocity, accelation, force
momentum, Toque
} - Vector.

Limit of a function:

A vector $\vec{v_0}$ is said to be limit of the vector function $\vec{f}(u)$ as u tends to u_0, if the limit of the scalar function $|\vec{f}(u) - \vec{v_0}|$, as u tends to u_0 is zero, that is if

$$\lim_{u \to u_0} |\vec{f}(u) - \vec{v_0}| = 0 \quad \text{or}$$

$$\lim_{u \to u_0} \vec{f}(u) = \vec{v_0}$$

Note:

If $\vec{f}(u) = f_1(u)\vec{i} + f_2(u)\vec{j} + f_3(u)\vec{k}$

then

$$\lim_{u \to u_0} \vec{f}(u) = \left\{\lim_{u \to u_0} f_1(u)\right\}\vec{i} + \left\{\lim_{u \to u_0} f_2(u)\right\}\vec{j} +$$

$$\left\{\lim_{u \to u_0} f_3(u)\right\}\vec{k}$$

Remark:

$$\lim_{u \to u_0} \left\{\vec{A}(u) + \vec{B}(u)\right\} = \lim_{u \to u_0} \vec{A}(u) + \lim_{u \to u_0} \vec{B}(u)$$

$$\lim_{u \to u_0} \left\{\vec{A}(u) - \vec{B}(u)\right\} = \lim_{u \to u_0} \vec{A}(u) - \lim_{u \to u_0} \vec{B}(u)$$

$$\lim_{u \to u_0} \left\{\vec{A}(u) \cdot \vec{B}(u)\right\} = \left\{\lim_{u \to u_0} \vec{A}(u)\right\} \cdot \left\{\lim_{u \to u_0} \vec{B}(u)\right\}$$

$$\lim_{u \to u_0} \left\{\vec{A}(u) \times \vec{B}(u)\right\} = \left\{\lim_{u \to u_0} \vec{A}(u)\right\} \times$$

$$\left\{\lim_{u \to u_0} \vec{B}(u)\right\}$$

Derivative of a vector function

A vector function $\vec{f}(u)$ is said to be derivative or differentiate with respect u if

$$\lim_{\Delta u \to 0} \frac{\vec{f}(u+\Delta u) - \vec{f}(u)}{\Delta u} \text{ exist}$$

This limit is called the derivative or differential coefficient of $\vec{f}(u)$ with respect u and is denoted by $\frac{d\vec{f}}{du}$

$$\frac{d}{du}\left(\frac{d\vec{f}}{du}\right) = \frac{d^2\vec{f}}{du^2}$$

$$\frac{d}{du}\left(\frac{d^2\vec{f}}{du^2}\right) = \frac{d^3\vec{f}}{du^3}$$

Note :

1. If $\vec{f}(u+\Delta u) = \vec{f}(u) + \Delta\vec{f}$, then

$$\vec{f}(u+\Delta u) - \vec{f}(u) = \Delta\vec{f}$$

$$\therefore \frac{d\vec{f}}{du} = \lim_{\Delta u \to 0} \frac{\Delta\vec{f}}{\Delta u}$$

2. If $\vec{f}(u)$ is constant vector then

$$\frac{d\vec{f}}{du} = 0$$

3. $\frac{d\vec{f}}{du}$ is rate of change of $\vec{f}$ to u.

Note:

1. Projection of $\vec{A}$ on $\vec{B}$ $= \frac{\vec{A} \cdot \vec{B}}{|\vec{B}|}$

Projection of $\vec{B}$ on $\vec{A}$ $= \frac{\vec{A} \cdot \vec{B}}{|\vec{A}|}$

2. $|\vec{A} \times \vec{B}|$ = area of parallelogram with $\vec{A}$ and $\vec{B}$ as adjusent sides.

Book work:

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i) If ϕ is a scalar functions of u and $\vec{a}$ a constant vector, than $\frac{d}{du}(\phi\vec{a}) = \vec{a}\frac{d\phi}{du}$

ii) If ϕ is a scalar function of u and $\vec{a}$ is a ~~ser~~ vector function of u, then,

$$\frac{d}{du}(\phi\vec{a}) = \frac{d\phi}{du}\vec{a} + \phi\frac{d\vec{a}}{du}$$

i) Proof:

$$\frac{d}{du}(\phi\vec{a}) = \lim_{\Delta u \to 0} \frac{\Delta(\phi\vec{a})}{\Delta u}$$

$$= \lim_{\Delta u \to 0} \frac{(\phi + \Delta\phi)\vec{a} - \phi(\vec{a})}{\Delta u}$$

$$= \lim_{\Delta u \to 0} \left(\frac{\phi\vec{a}}{\Delta u} + \frac{\Delta\phi\,\vec{a}}{\Delta u} - \frac{\phi\vec{a}}{\Delta u} \right)$$

$$= \lim_{\Delta u \to 0} \frac{\Delta \phi}{\Delta u} \vec{a}$$

$$= \frac{d\phi}{du} \vec{a}$$

$$\frac{d}{du}(\phi \vec{a}) = \vec{a} \frac{d\phi}{du}$$

ii) Proof:

$$\frac{d}{du}(\phi \vec{a}) = \lim_{\Delta u \to 0} \frac{(\phi + \Delta\phi)(\vec{a} + \Delta\vec{a}) - \phi \vec{a}}{\Delta u}$$

$$= \lim_{\Delta u \to 0} \left(\frac{\phi \vec{a}}{\Delta u} + \frac{\phi \Delta \vec{a}}{\Delta u} + \frac{\Delta\phi \vec{a}}{\Delta u} + \frac{\Delta\phi \Delta\vec{a}}{\Delta u} - \frac{\phi \vec{a}}{\Delta u} \right)$$

$$= \lim_{\Delta u \to 0} \frac{\phi \Delta \vec{a}}{\Delta u} + \lim_{\Delta u \to 0} \frac{\Delta\phi \vec{a}}{\Delta u} + \lim_{\Delta u \to 0} \frac{\Delta\phi \Delta\vec{a}}{\Delta u}$$

$$= \frac{d\phi}{du} \vec{a} + \frac{d\vec{a}}{du} \phi + 0$$

$$= \frac{d\phi}{du} \vec{a} + \frac{d\vec{a}}{du} \phi .$$

$$\frac{d}{du}(\phi \vec{a}) = \frac{d\phi}{du} \vec{a} + \phi \frac{d\vec{a}}{du}$$

Book work:

i) $\frac{d}{du}(\vec{A} \pm \vec{B}) = \frac{d\vec{A}}{du} \pm \frac{d\vec{B}}{du}$

ii) $\frac{d}{du}(\vec{A}.\vec{B}) = \frac{d\vec{A}}{du}.\vec{B} + \vec{A}.\frac{d\vec{B}}{du}$

iii) $\frac{d}{du}(\vec{A}\times\vec{B}) = \frac{d\vec{A}}{du}\times\vec{B} + \vec{A}\times\frac{d\vec{B}}{du}$

~~iv) $\frac{d}{du}$ $\vec{A}\cdot\vec{B}\cdot B$~~

iv) $\frac{d}{du}[\vec{A}, \vec{B}\,\vec{C}] = \left[\frac{d\vec{A}}{du}.\vec{B}.\vec{C}\right] + \left[\frac{d\vec{B}}{du}.\vec{A}.\vec{C}\right]$

$\left[\vec{A}.\vec{B}.\frac{d\vec{C}}{du}\right]$

v) $\frac{d}{du}\left[\vec{A}\times(\vec{B}\times\vec{C})\right] = \frac{d\vec{A}}{du}\times(\vec{B}\times\vec{C}) +$

$\vec{A}\times\left(\frac{d\vec{B}}{du}\times\vec{C}\right) + \vec{A}\times\left(\vec{B}\times\frac{d\vec{C}}{du}\right)$

i) Proof:

Method - 1

$$\frac{d}{du}(\vec{A}+\vec{B}) = \lim_{\Delta u\to 0}\frac{(\vec{A}+\Delta\vec{A}) + (\vec{B}+\Delta\vec{B}) - (\vec{A}+\vec{B})}{\Delta u}$$

$$= \lim_{\Delta u\to 0}\left[\frac{\vec{A}}{\Delta u} + \frac{\Delta\vec{A}}{\Delta u} + \frac{\vec{B}}{\Delta u} + \frac{\Delta\vec{B}}{\Delta u} - \frac{\vec{A}}{\Delta u} - \frac{\vec{B}}{\Delta u}\right]$$

$$= \lim_{\Delta u\to 0}\frac{\Delta\vec{A}}{\Delta u} + \lim_{\Delta u\to 0}\frac{\Delta\vec{B}}{\Delta u}$$

$$\frac{d}{du}(\vec{A}+\vec{B}) = \frac{d\vec{A}}{du} + \frac{d\vec{B}}{du}$$

Method - 2.

$\vec{A} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$

$\vec{B} = b_1\vec{i} + b_2\vec{j} + b_3\vec{k}$

$$\frac{d}{du}(\vec{A}+\vec{B}) = \frac{d}{du}\left[(a_1+b_1)\vec{i} + (a_2+b_2)\vec{j} + (a_3+b_3)\vec{k}\right]$$

$$= \frac{d}{du}(a_1+b_1)\vec{i} + \frac{d}{du}(a_2+b_2)\vec{j} + \frac{d}{du}(a_3+b_3)\vec{k}$$

$$= \left(\frac{da_1}{du} + \frac{db_1}{du}\right)\vec{i} + \left(\frac{da_2}{du} + \frac{db_2}{du}\right)\vec{j} + \left(\frac{da_3}{du} + \frac{db_3}{du}\right)\vec{k}$$

$$= \left(\frac{da_1}{du}\vec{i} + \frac{da_2}{du}\vec{j} + \frac{da_3}{du}\vec{k}\right) + \left(\frac{db_1}{du}\vec{i} + \frac{db_2}{du}\vec{j} + \frac{db_3}{du}\vec{k}\right)$$

$$\frac{d}{du}(\vec{A}+\vec{B}) = \frac{d\vec{A}}{du} + \frac{d\vec{B}}{du}$$

ii) Proof:

$$\frac{d}{du}(\vec{A}\cdot\vec{B}) = \lim_{\Delta u \to 0} \frac{(\vec{A}+\Delta\vec{A})\cdot(\vec{B}+\Delta\vec{B}) - \vec{A}\cdot\vec{B}}{\Delta u}$$

$$= \lim_{\Delta u \to 0}\left(\frac{\vec{A}\cdot\vec{B}}{\Delta u} + \frac{\vec{A}\cdot\Delta\vec{B}}{\Delta u} + \frac{\Delta\vec{A}\cdot\vec{B}}{\Delta u} + \frac{\Delta\vec{A}\cdot\Delta\vec{B}}{\Delta u} - \frac{\vec{A}\cdot\vec{B}}{\Delta u}\right)$$

$$= \lim_{\Delta u \to 0}\frac{\vec{A}\cdot\Delta\vec{B}}{\Delta u} + \lim_{\Delta u \to 0}\frac{\Delta\vec{A}\cdot\vec{B}}{\Delta u} + 0$$

$$= \frac{d\vec{B}}{du}\cdot\vec{A} + \frac{d\vec{A}}{du}\cdot\vec{B}$$

$$\frac{d}{du}(\vec{A}\cdot\vec{B}) = \vec{A}\cdot\frac{d\vec{B}}{du} + \frac{d\vec{A}}{du}\cdot\vec{B}$$

iii) Prof:

$$\frac{d}{du}(\vec{A}\times\vec{B}) = \lim_{\Delta u\to 0}\frac{(\vec{A}+\Delta\vec{A})\times(\vec{B}+\Delta\vec{B}) - \vec{A}\times\vec{B}}{\Delta u}$$

$$= \lim_{\Delta u\to 0}\left(\frac{\vec{A}\times\vec{B}}{\Delta u} + \frac{\vec{A}\times\Delta\vec{B}}{\Delta u} + \frac{\Delta\vec{A}\times\vec{B}}{\Delta u} + \frac{\Delta\vec{A}\times\Delta\vec{B}}{\Delta u} - \frac{\vec{A}\times\vec{B}}{\Delta u}\right)$$

$$= \lim_{\Delta u\to 0}\frac{\vec{A}\times\Delta\vec{B}}{\Delta u} + \lim_{\Delta u\to 0}\frac{\Delta\vec{A}\times\vec{B}}{\Delta u} + 0$$

$$\frac{d}{du}(\vec{A}\times\vec{B}) = \vec{A}\times\frac{d\vec{B}}{du} + \frac{d\vec{A}}{du}\times\vec{B}$$

Note :

1. If $\vec{f}(u)$ is a vector function of the scalar u, where u is a function of t, then,

$$\cancel{\frac{d\vec{f}}{dt}} \quad \frac{d\vec{f}}{dt} = \frac{d\vec{f}}{du}\cdot\frac{du}{dt}$$

2. If $\vec{r}$ is position ~~vector~~ vector of P, then,

$$\frac{d\vec{r}}{dt} = \text{velocity of } P$$

Problem - 1

If $\vec{A} = u\vec{i} + u^2\vec{j} + u^3\vec{k}$, $\vec{B} = u^3\vec{i} + u^2\vec{j} + u\vec{k}$

find (i) $\dfrac{d(\vec{A} \cdot \vec{B})}{du}$ (ii) $\dfrac{d(\vec{A} \times \vec{B})}{du}$

Soln:

i) $\vec{A} \cdot \vec{B} = (u\vec{i} + u^2\vec{j} + u^3\vec{k}) \cdot (u^3\vec{i} + u^2\vec{j} + u\vec{k})$

$= u^4 + u^4 + u^4$

$\vec{A} \cdot \vec{B} = 3u^4$

$$\frac{d}{du}(\vec{A} \cdot \vec{B}) = \frac{d}{du}(3u^4) = 3 \times 4u^3 = 12u^3$$

~~ii) $d(\vec{A} \times \vec{B})$~~

ii) $$\vec{A} \times \vec{B} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ u & u^2 & u^3 \\ u^3 & u^2 & u \end{vmatrix}$$

$$\vec{A} \times \vec{B} = \vec{i}(u^3 - u^5) - \vec{j}(u^2 - u^6) + \vec{k}(u^3 - u^5)$$

$$\frac{d(\vec{A} \times \vec{B})}{du} = \vec{i}(3u^2 - 5u^4) - \vec{j}(2u - 6u^5) + \vec{k}(3u^2 - 5u^4)$$

$$\therefore \frac{d(\vec{A} \times \vec{B})}{du} = (3u^2 - 5u^4)\vec{i} - (2u - 6u^5)\vec{j} + (3u^2 - 5u^4)\vec{k}$$

Problem - 2

$\frac{d}{du}(\vec{A}\cdot\vec{B})$, $\frac{d}{du}(\vec{A}\times\vec{B})$ is

$\vec{A} = \vec{i} + u\vec{j} + u^2\vec{k}$, $\vec{B} = u^2\vec{i} - u\vec{j} + \vec{k}$

Soln:

i) $\vec{A}\cdot\vec{B} = (\vec{i} + u\vec{j} + u^2\vec{k})\cdot(u^2\vec{i} - u\vec{j} + \vec{k})$

$= u^2 - u^2 + u^2 = u^2$

$\frac{d}{du}(\vec{A}\cdot\vec{B}) = \frac{d}{du}(u^2) = 2u$

ii) $\vec{A}\times\vec{B} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & u & u^2 \\ u^2 & -u & 1 \end{vmatrix}$

$= \vec{i}(u+u^3) - \vec{j}(1-u^4) + \vec{k}(-u-u^3)$

$\cancel{= (u+u^3)\vec{i} - \vec{j}(1-u^4)\vec{j} - \vec{k}}$

$\vec{A}\times\vec{B} = (u+u^3)\vec{i} - (1-u^4)\vec{j} - (u+u^3)\vec{k}$

$\frac{d}{du}(\vec{A}\times\vec{B}) = (1+3u^2)\vec{i} - (-4u^3)\vec{j} - (1+3u^2)\vec{k}$

$\frac{d}{du}(\vec{A}\times\vec{B}) = (1+3u^2)\vec{i} + (4u^3)\vec{j} - (1+3u^2)\vec{k}$

Problem - 3

Find $\frac{d}{du}(\vec{A}\times\vec{B})$ in the following cases.

i) $\vec{A} = 2u\vec{i} + u^2\vec{j}$, $\vec{B} = -u\vec{j} + \vec{k}$

ii) $\vec{A} = 5u^2\vec{i} + u\vec{j} - u^3\vec{k}$, $\vec{B} = \sin u\,\vec{i} - \cos u\,\vec{j}$

Soln:

i) $$\vec{A} \times \vec{B} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2u & u^2 & 0 \\ 0 & -u & 1 \end{vmatrix}$$

$$= \vec{i}(u^2+0) - \vec{j}(2u+0) + \vec{k}(-2u^2+0)$$

$$\vec{A} \times \vec{B} = u^2\vec{i} - 2u\vec{j} - 2u^2\vec{k}$$

$$\frac{d}{du}(\vec{A} \times \vec{B}) = 2u\vec{i} - 2\vec{j} - 4u\vec{k}$$

ii) $$\vec{A} \times \vec{B} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 5u^2 & u & -u^3 \\ \sin u & -\cos u & 0 \end{vmatrix}$$

$$= \vec{i}(0 - u^3\cos u) - \vec{j}(0 + u^3\sin u) + \vec{k}(-5u^2\cos u - u\sin u)$$

$$(\vec{A} \times \vec{B}) = -u^3\cos u\,\vec{i} - u^3\sin u\,\vec{j} - (5u^2\cos u + u\sin u)\vec{k}$$

$$\frac{d}{du}(\vec{A} \times \vec{B}) = (u^3\sin u - 3u^2\cos u)\vec{i} - (u^3\cos u + 3u^2\sin u)\vec{j} - \left[(5u^2(-\sin u) + 10u\cos u) + (u\cos u + \sin u)\right]\vec{k}$$

$$= (u^3 \sin u - 3u^2 \cos u)\vec{i} - (u^3 \cos u + 3u^2 \sin u)\vec{j} - \left[(10u \cos u - 5u^2 \sin u) + (u \cos u + \sin u)\right]\vec{k}$$

$$\frac{d}{du}(\vec{A} \times \vec{B}) = (u^3 \sin u - 3u^2 \cos u)\vec{i} - (u^3 \cos u + 3u^2 \sin u) - \left[(11 u \cos u + \sin u - 5u^2 \sin u)\right]\vec{k}$$

/8/2020 Problem - 4

Show that the necessary and sufficient for the non-zero vector function $\vec{f}(u)$ to be of constant magnitude is $\vec{f} \cdot \frac{d\vec{f}}{du} = 0$ (i.e $\vec{f}$ and $\frac{d\vec{f}}{du}$ are perpendicular to each other).

Solution:

Let $\vec{f} = f \cdot \hat{f}$

Then $\vec{f} \cdot \vec{f} = (f \cdot \hat{f}) \cdot (f \cdot \hat{f})$

$= f^2 (\hat{f} \cdot \hat{f})$

$= f^2 (1)$

$$\vec{f} \cdot \vec{f} = f^2 \longrightarrow ①$$

Differentiate ① with respect to u, we get

$$\frac{d}{du}(\vec{f} \cdot \vec{f}) = \frac{d}{du}(f^2)$$

$$\frac{d\vec{f}}{du} \cdot \vec{f} + \vec{f} \cdot \frac{d\vec{f}}{du} = 2f \frac{df}{du}$$

$$\vec{f}\cdot\frac{d\vec{f}}{du}+\vec{f}\cdot\frac{d\vec{f}}{du}=2f\frac{df}{du}$$

$$2\vec{f}\cdot\frac{d\vec{f}}{du}=2f\frac{df}{du}$$

$$\vec{f}\cdot\frac{d\vec{f}}{du}=f\frac{df}{du}\longrightarrow ②$$

necessary part:

Let $|\vec{f}(u)|$ be constant.

(i.e) f is constant.

$$\therefore \frac{df}{du}=0$$

From ②, we get,

$$\vec{f}\cdot\frac{d\vec{f}}{du}=0$$

Sufficient part:

Let $\vec{f}\cdot\frac{d\vec{f}}{du}=0$

From ②, we get,

$$f\frac{df}{du}=0$$

$$\Rightarrow \frac{df}{du}=0 \qquad [\because f\neq 0]$$

f is constant

(i.e) magnitude of $\vec{f}(u)$ is constant.

Problem - 5

Show that if $\vec{f}$ is not of constant direction then $\left|\frac{d\vec{f}}{du}\right| \neq \frac{d}{du}|\vec{f}|$

Solution:

Let $\vec{f} = f \cdot \hat{f}$

$$\text{then } \vec{f} \cdot \vec{f} = (f \cdot \hat{f}) \cdot (f \cdot \hat{f})$$

$$= f^2 \cdot (\hat{f} \cdot \hat{f})$$

$$= f^2 (1)$$

$$\vec{f} \cdot \vec{f} = f^2 \longrightarrow ①$$

Differentiating ① with respect to u, we get,

$$\frac{d}{du}(\vec{f} \cdot \vec{f}) = \frac{d}{du}(f^2)$$

$$\frac{d\vec{f}}{du} \cdot \vec{f} + \vec{f} \cdot \frac{d\vec{f}}{du} = 2f \frac{df}{du}$$

$$\vec{f} \cdot \frac{d\vec{f}}{du} + \vec{f} \cdot \frac{d\vec{f}}{du} = 2f \frac{df}{du}$$

$$2\vec{f} \cdot \frac{d\vec{f}}{du} = 2f \frac{df}{du}$$

$$\vec{f} \cdot \frac{d\vec{f}}{du} = f \frac{df}{du}$$

$$f \frac{df}{du} = |\vec{f}| \left|\frac{d\vec{f}}{du}\right| \cos\theta \longrightarrow ②$$ when θ is angle between $\vec{f}$ and $\frac{d\vec{f}}{du}$

Given $\vec{f}$ is not constant direction.

$\therefore$ $\vec{f}$ and $\frac{d\vec{f}}{du}$ are not parallel.

Hence $\theta \neq 0$

$$③ \Rightarrow f \frac{df}{du} = \vec{f} \left| \frac{d\vec{f}}{du} \right| \cos\theta$$

$$\frac{df}{du} = \left| \frac{d\vec{f}}{du} \right| \cos\theta$$

$$\cancel{\phi}\ \theta \neq 0 \Rightarrow \frac{df}{du} \neq \left| \frac{d\vec{f}}{du} \right|$$

$$\therefore \frac{d}{du}(\vec{f}) \neq \left| \frac{d\vec{f}}{du} \right|$$

Problem-6

Show the necessary and sufficient condition for the vector funtion $\vec{f}(u)$ is have a constant direction is $\vec{f} \times \frac{d\vec{f}}{du} = 0$. ((i.e) $\vec{f}$ and $\frac{d\vec{f}}{du}$ are parallel)

Solution:

Let $\vec{f} = f \cdot \hat{f}$

Then $$\frac{d\vec{f}}{du} = f \cdot \frac{d\hat{f}}{du} + \frac{df}{du} \cancel{\phi} \hat{f}$$

$\cancel{f} \times$

$$\vec{f} \times \frac{d\vec{f}}{du} = \vec{f} \times \left[f \frac{d\hat{f}}{du} + \frac{df}{du} \hat{f} \right]$$

$$= f \left(\vec{f} \times \frac{d\hat{f}}{du} \right) + \vec{f} \times r$$

$$= f\left(\vec{f} \times \frac{d\hat{f}}{du}\right) + \frac{df}{du}(\vec{f} \times \hat{f})$$

$$= f\left(f\hat{f} \times \frac{d\hat{f}}{du}\right) + \frac{d\vec{f}}{du}(f\hat{f} \times \hat{f})$$

$$= f^2\left(\hat{f} \times \frac{d\hat{f}}{du}\right) + f\frac{d\vec{f}}{du}(\hat{f} \times \hat{f})$$

$$= f^2\left(\hat{f} \times \frac{d\hat{f}}{du}\right) + 0$$

$$\vec{f} \times \frac{d\vec{f}}{du} = f^2\left(\hat{f} * \frac{d\hat{f}}{du}\right) \longrightarrow ①$$

necessary part:

Let $\vec{f}(u)$ have constant direction.

Then $\hat{f}$ is also in constant direction.

$$\therefore \frac{d\vec{f}}{du} = 0$$

from ① we have

$$\vec{f} \times \frac{d\vec{f}}{du} = 0$$

Sufficient part:

$$\text{let } \vec{f} \times \frac{d\vec{f}}{du} = 0$$

$$\text{i.e } f\hat{f} \times \frac{d}{du}(f\hat{f}) = 0$$

$$\Rightarrow f^2\left(\hat{f} \times \frac{d\hat{f}}{du}\right) = 0$$

$$\Rightarrow \hat{f} \times \frac{d\hat{f}}{du} = 0 \quad \{\because f \neq 0\}$$

$\Rightarrow \hat{f}$ and $\frac{d\hat{f}}{du}$ are parallel or $\frac{d\vec{f}}{du} = 0$

But $\vec{f}$ and $\frac{d\vec{f}}{du}$ are not parallel.

Because they are perpendicular.

Hence $\frac{d\vec{f}}{du} = 0$

$\Rightarrow \vec{f}$ is of constant direction.

$\Rightarrow \vec{f}$ is of constant direction

Problem - 7

Show that the necessary and sufficient condition for a ~~f~~ vector function $\vec{f}(u)$ may be constant (i.e) $\frac{d\vec{f}}{du} = 0$.

Solution:

Let $\vec{f} = f_1\vec{i} + f_2\vec{j} + f_3\vec{k}$

$$\frac{d\vec{f}}{du} = \frac{df_1}{du}\vec{i} + \frac{df_2}{du}\vec{j} + \frac{df_3}{du}\vec{k} = 0$$

$$\Rightarrow \frac{df_1}{du} = 0, \quad \frac{df_2}{du} = 0, \quad \frac{df_3}{du} = 0$$

$\Rightarrow f_1, f_2, f_3$ are constants.

$\Rightarrow \vec{f}$ is a constant.

Problem - 8

If $\vec{f}$ and $\vec{g}$ are vector functions of u such that $\vec{f} \times \frac{d\vec{g}}{du} = \vec{g} \times \frac{d\vec{f}}{du}$ for all values of u, show that $\vec{f}$ and $\vec{g}$ are always perpendicular to a fixed direction.

Solution:

$$\vec{f} \times \frac{d\vec{g}}{du} = \vec{g} \times \frac{d\vec{f}}{du}$$

$$\Rightarrow \vec{f} \times \frac{d\vec{g}}{du} - \vec{g} \times \frac{d\vec{f}}{du} = 0$$

$$\Rightarrow \vec{f} \times \frac{d\vec{g}}{du} + \frac{d\vec{f}}{du} \times \vec{g} = 0$$

$$\Rightarrow \frac{d}{du}(\vec{f} \times \vec{g}) = 0$$

$\Rightarrow$ $\vec{f} \times \vec{g}$ is a constant vector.

$\Rightarrow$ $\vec{f} \times \vec{g}$ is in a fixed directions.

By definition of cross product $\vec{f} \times \vec{g}$ is perpendicular to in fixed direction.

10/8/2020

Problem - 9

Problem 12 [illegible]

If $\vec{r} = \vec{a}\cos\omega t + \vec{b}\sin\omega t$, where $\vec{a}, \vec{b}$ are constant vectors and ω, a constant scalor. Show that $\vec{r} \times \frac{d\vec{r}}{dt} = \omega(\vec{a} \times \vec{b})$

$$\frac{d^2\vec{r}}{dt^2} = -\omega^2\vec{r}.$$

Solution:

$$\vec{r} = \vec{a}\cos\omega t + \vec{b}\sin\omega t \rightarrow ①$$

Differentiate ① with respect to t, we have

$$\frac{d\vec{r}}{dt} = \vec{a}(-\sin\omega t).\omega + \vec{b}\cos\omega t.\omega$$

~~$\omega(-a$~~

$$\frac{d\vec{r}}{dt} = \omega(-\vec{a}\sin\omega t + \vec{b}\cos\omega t) \longrightarrow ②$$

$$\vec{r} \times \frac{d\vec{r}}{dt} = (\vec{a}\cos\omega t + \vec{b}\sin\omega t) \times \omega(-\vec{a}\sin\omega t + \vec{b}\cos\omega t)$$

x ,

~~$= \omega[-\cos\omega t\sin\omega t\,\vec{a} \times \vec{a} + \vec{a} \times \vec{b}\cos^2\omega t + -\vec{a} \times \vec{b}\sin^2\omega t + \sin\omega t\cos\omega t\,\vec{b} \times \vec{b}]$~~

~~$= \omega[\vec{a} \times \vec{b}\cos^2\omega t -$~~

$$= \omega\Big[-\cos\omega t \ \sin\omega t \ \vec{a}\times\vec{a} + \vec{a}\times\vec{b}\cos^2\omega t$$

$$- \vec{b}\times\vec{a}\ \sin^2\omega t + \sin\omega t \cos\omega t \ \vec{b}\times\vec{b}\Big]$$

$$= \omega\left[\vec{a}\times\vec{b}\cos^2\omega t + \vec{a}\times\vec{b}\sin^2\omega t\right]$$

$$= \omega\left[\vec{a}\times\vec{b}\right]\left[\cos^2\omega t + \sin^2\omega t\right]$$

$$= \omega(\vec{a}\times\vec{b})$$

Differentiate ② with respect to t,

$$\frac{d^2\vec{r}}{dt^2} = -\vec{a}\cos\omega t\,.\,\omega^2 + \vec{b}(-\sin\omega t).\,\omega^2$$

$$= -\omega^2\left[\vec{a}\cos\omega t + \vec{b}\sin\omega t\right]$$

$$\frac{d^2\vec{r}}{dt^2} = -\omega^2\vec{r} \qquad [\because \text{by } ①]$$

$$\therefore \vec{r}\times\frac{d\vec{r}}{dt} = \omega(\vec{a}\times\vec{b})$$

$$\frac{d^2\vec{r}}{dt^2} = -\omega^2\vec{r}$$

Problem - 10

If $\vec{a}, \vec{b}, \vec{w}$ are vector functions a scalor variable u and if $\frac{d\vec{a}}{du} = \vec{w}\times\vec{a}$ $\frac{d\vec{b}}{du} = \vec{w}\times\vec{b}$. Then, S.T $\frac{d}{du}(\vec{a}\times\vec{b}) = \vec{w}\times(\vec{a}\times\vec{b})$

Solution:

$$\frac{d}{du}(\vec{a}\times\vec{b}) = \frac{d\vec{a}}{du}\times\vec{b} + \vec{a}\times\frac{d\vec{b}}{du}$$

$$= (\vec{\omega}\times\vec{a})\times\vec{b} + \vec{a}\times(\vec{\omega}\times\vec{b})$$

$$= (\vec{\omega}\cdot\vec{b})\vec{a} - (\vec{a}\cdot\vec{b})\vec{\omega} +$$

$$(\vec{a}\cdot\vec{b})\vec{\omega} - (\vec{a}\cdot\vec{\omega})\vec{b}$$

$$= (\vec{\omega}\cdot\vec{b})\vec{a} - (\vec{a}\cdot\vec{\omega})\vec{b}$$

$$\frac{d}{du}(\vec{a}\times\vec{b}) = \vec{\omega}\times(\vec{a}\times\vec{b})$$

Directional Derivative

Suppose $\phi(x, y, z)$ is a scalar point function and $\phi(P)$ is the variable of ϕ at P. P' is any point close to P, then the limit.

$$\lim_{P' \to P} \frac{\phi(P') - \phi(P)}{PP'} \text{ (or) } \lim_{PP' \to 0} \frac{\phi(P') - \phi(P)}{PP'}$$

is called directional derivative of ϕ.

(i) at the point P.

(ii) in the direction from P to P'

The operator ∇ : (Del or nabla)

$$\nabla = \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}$$

Gradient:

The gradient of a scalar point function $\phi(x, y, z)$ denoted by ~~gra~~ grad(ϕ) or $\nabla\phi$ is defined as

~~$grad\ \phi = \vec{i}\frac{d\phi}{dx} + \vec{j}\frac{d\phi}{dy} + \vec{k}\frac{d\phi}{dz}$~~

~~$= i\frac{d}{}$~~

$$grad(\phi) = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$$

$$= \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\phi$$

$$= \nabla\phi \rightarrow \text{gradient}$$

Formula.

1. Normal to the surface $\phi(x, y, z) = \nabla\phi$
2. Unit normal to the surface $= \frac{\nabla\phi}{|\nabla\phi|}$
3. Directional Derivative of ϕ along $\bar{e} = \nabla\phi \cdot \hat{e}$
4. Directional Derivative is maximum along $\nabla\phi$
5. Maximum Direction Derivative $= \phi = |\nabla\phi|$
6. Angle between two surfaces $\phi_1 = c_1$ and $\phi_2 = c_2$ is $\cos\theta = \frac{\nabla\phi_1 \cdot \nabla\phi_2}{|\nabla\phi_1|\,|\nabla\phi_2|}$

Problem - 11

Find $\nabla\phi$ if

i) $\phi = x^2 y^3 z^2$ (ii) $\phi = xyz - x^2$

Solution:

i) $$\nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$$

$$= \vec{i}\frac{\partial}{\partial x}(x^2y^3z^2) + \vec{j}\frac{\partial}{\partial y}(x^2y^3z^2) + \vec{k}\frac{\partial}{\partial z}(x^2y^3z^2)$$

$$\nabla\phi = \vec{i}(2xy^3z^2) + \vec{j}(3x^2y^2z^2) + \vec{k}(2x^2y^3z)$$

(ii) $$\nabla\phi = \vec{i}\frac{\partial}{\partial x}(xyz - x^2) + \vec{j}\frac{\partial}{\partial y}(xyz - x^2) + \vec{k}\frac{\partial}{\partial z}(xyz - x^2)$$

$$\nabla\phi = \vec{i}(yz - 2x) + \vec{j}(xz) + \vec{k}(xy)$$

12/8/2020 Problem - 12

Find $\nabla\phi$ in the following cases at the points specified

i) $\phi(x, y, z) = 2xz - y^2$ at $(1, 3, 2)$

ii) $\phi(x, y, z) = x + xy^2 + yz^2$ at $(1, 0, 0)$

iii) $\phi(x, y, z) = y^2(x - z)$ at $(1, 1, 2)$

Solution

i) $\nabla\phi = \vec{i}\dfrac{\partial\phi}{\partial x} + \vec{j}\dfrac{\partial\phi}{\partial y} + \vec{k}\dfrac{\partial\phi}{\partial z}$

$= \vec{i}\dfrac{\partial}{\partial x}(2xz - y^2) + \vec{j}\dfrac{\partial}{\partial y}(2xz - y^2) +$

$\vec{k}\dfrac{\partial}{\partial z}(2xz - y^2)$

$= \vec{i}(2z) + \vec{j}(-2y) + \vec{k}(2x)$

$\nabla\phi = (2z)\vec{i} - (2y)\vec{j} + (2x)\vec{k}$

~~$(\nabla\phi)_{(1,3,2)} = (2)(2)\vec{i} - (2$~~

$(\nabla\phi)_{(1,3,2)} = 4\vec{i} - 6\vec{j} + 2\vec{k}$

ii) $\nabla\phi = \vec{i}\dfrac{\partial}{\partial x}(x + xy^2 + yz^2) + \vec{j}\dfrac{\partial}{\partial y}(x + xy^2 + yz^2)$

$+ \vec{k}\dfrac{\partial}{\partial z}(x + xy^2 + yz^2)$

$\nabla\phi = \vec{i}(1 + y^2) + \vec{j}(2xy + z^2) + \vec{k}(2yz)$

$(\nabla\phi)_{(1,0,0)} = \vec{i}(1+0) + \vec{j}(0) + \vec{k}(0)$

$(\nabla\phi)_{(1,0,0)} = \vec{i}$

iii) $\nabla\phi = \vec{i}\dfrac{\partial}{\partial x}(y^2(x-z)) + \vec{j}\dfrac{\partial}{\partial y}(y^2(x-z)) +$

$\vec{k}\dfrac{\partial}{\partial z}(y^2(x-z))$

$= \vec{i}\dfrac{\partial}{\partial x}(y^2x - y^2z) + \vec{j}\dfrac{\partial}{\partial y}(y^2x - y^2z) +$

$\vec{k}\dfrac{\partial}{\partial z}(y^2x - y^2z)$

$= \vec{i}(y^2) + \vec{j}(2xy - 2yz) + \vec{k}(-y^2)$

$$\nabla\phi = (y^2)\vec{i} + (2xy - 2yz)\vec{j} - (y^2)\vec{k}$$

$$(\nabla\phi)_{(1,1,2)} = (1)\vec{i} + ((2)(1)(1) - 2(1)(2))\vec{j} - (1)\vec{k}$$

$$(\nabla\phi)_{(1,1,2)} = \vec{i} - 2\vec{j} - \vec{k}$$

~~Problem – 13~~

~~Find the directional derivative $\phi = x + x^2y^2 + yz^3$ at $(0,1,1)$ in the direction of the vector $2\vec{i} + 2\vec{j} - \vec{k}$~~

~~Solution:~~

~~$\phi = x + x^2y^2 + yz^3$~~

~~$\nabla\phi = \vec{i}\frac{\partial}{\partial x}(x + x^2y^2 + yz^3) + \cancel{\vec{j}\phi}$~~

~~$\vec{j}\frac{\partial}{\partial y}(x + x^2y^2 + yz^3) + \vec{k}\frac{\partial}{\partial z}(x + x^2y^2 + yz^3)$~~

~~$\nabla\phi = \vec{i}(1 + 2xy) + \vec{j}(2xy + z^3) + \vec{k}(3yz^2)$~~

~~$\nabla\phi(0,1,1) = (1+0)\vec{i} + (0+1)\vec{k}$~~

Problem – 13

Find the directional derivative $\phi = x + xy^2 + yz^3$ at $(0,1,1)$ in the direction of the vector $2\vec{i} + 2\vec{j} - \vec{k}$

Solution:

$$\phi = x + xy^2 + yz^3$$

$$\nabla\phi = \vec{i}\,\frac{\partial}{\partial x}(x + xy^2 + yz^3) + \vec{j}\,\frac{\partial}{\partial y}(x + xy^2 + yz^3)$$

$$+ \vec{k}\,\frac{\partial}{\partial z}(x + xy^2 + yz^3)$$

$$\nabla\phi = \vec{i}(1 + y^2) + \vec{j}(2xy + z^3) + \vec{k}(3yz^2)$$

$$\nabla\phi_{(0,1,1)} = (1+1)\vec{i} + (0+1)\vec{j} + 3\vec{k}$$

$$\nabla\phi_{(0,1,1)} = 2\vec{i} + \vec{j} + 3\vec{k}$$

$$\vec{e} = 2\vec{i} + 2\vec{j} - \vec{k}$$

$$\hat{e} = \frac{\vec{e}}{|\vec{e}|} = \frac{2\vec{i} + 2\vec{j} - \vec{k}}{\sqrt{4+4+1}} = \frac{2\vec{i} + 2\vec{j} - \vec{k}}{\sqrt{9}}$$

$$\hat{e} = \frac{2\vec{i} + 2\vec{j} - \vec{k}}{3} = \frac{1}{3}(2\vec{i} + 2\vec{j} - \vec{k})$$

Directional derivative of ϕ along $\vec{e}$ $\Big\}$ $= \nabla\phi \cdot \hat{e}$

$$= (2\vec{i} + \vec{j} + 3\vec{k}) \cdot \frac{1}{3}(2\vec{i} + 2\vec{j} - \vec{k})$$

$$= \frac{1}{3}(4 + 2 - 3)$$

$$= \frac{1}{3}(3)$$

Directional derivative = 1

Problem - 14

Find the directional derivative of $\phi = x^2yz + 4xz^2$ at $(1, -2, -1)$ along $2\vec{i} - \vec{j} - 2\vec{k}$.

Solution:

$$\phi = x^2yz + 4xz^2$$

$$\nabla\phi = \vec{i}\,\frac{\partial\phi}{\partial x} + \vec{j}\,\frac{\partial\phi}{\partial y} + \vec{k}\,\frac{\partial\phi}{\partial z}$$

$$= \vec{i}\left(2xyz + 4z^2\right) + \vec{j}\left(x^2z\right) + \vec{k}\left(x^2y + 8xz\right)$$

$$\nabla\phi_{(1,-2,-1)} = \vec{i}\left(2(1)(-2)(-1) + 4(1)\right) + \vec{j}(1)(-1)$$

$$+ \vec{k}\left((1)(-2) + 8(1)(-1)\right)$$

$$= \vec{i}\left(4 + 4\right) + \vec{j}(-1) + \vec{k}\left(-2 - 8\right)$$

$$\nabla\phi\,(1,-2,-1) = 8\vec{i} - \vec{j} - 10\vec{k}$$

$$\vec{e} = 2\vec{i} - \vec{j} - 2\vec{k}$$

$$\hat{e} = \frac{\vec{e}}{|\vec{e}|} = \frac{2\vec{i} - \vec{j} - 2\vec{k}}{\sqrt{4+1+4}} = \frac{2\vec{i} - \vec{j} - 2\vec{k}}{\sqrt{9}}$$

$$\hat{e} = \frac{1}{3}\left(2\vec{i} - \vec{j} - 2\vec{k}\right)$$

$$\left.\begin{array}{l}\text{Directional derivative of} \\ \phi \text{ along } \vec{e}\end{array}\right\} = \nabla\phi \cdot \hat{e}$$

$$= \left(8\vec{i} - \vec{j} - 10\vec{k}\right) \cdot \frac{1}{3}\left(2\vec{i} - \vec{j} - 2\vec{k}\right)$$

$$= \frac{1}{3}\left(16 + 1 + 20\right)$$

$$= \frac{1}{3}(37)$$

Directional derivative $= \frac{37}{3}$

Problem - 15

Find the directional derivative of the function $x^2+y^2+z^2$ at $(3, 6, 9)$ in the direction whose d.c's are $1/3, 2/3, 2/3$.

Solution:

$$\hat{e} = \frac{1}{3}\vec{i} + \frac{2}{3}\vec{j} + \frac{2}{3}\vec{k}$$

$$\phi = x^2+y^2+z^2$$

$$\nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$$

$$\nabla\phi = \vec{i}(2x) + \vec{j}(2y) + \vec{k}(2z)$$

$$\nabla\phi_{(3,6,9)} = 6\vec{i} + 12\vec{j} + 18\vec{k}$$

$$\left.\begin{array}{l}\text{Directional derivative}\\ \text{of } \phi \text{ along } \vec{e}\end{array}\right\} = \nabla\phi \cdot \hat{e}$$

$$= (6\vec{i} + 12\vec{j} + 18\vec{k}) \cdot \left(\frac{1}{3}\vec{i} + \frac{2}{3}\vec{j} + \frac{2}{3}\vec{k}\right)$$

$$= 2 + 8 + 12$$

Directional derivation $= 22$

13/8/2020

Problem - 16

$$\nabla\phi = (y+y^2+z^2)\vec{i} + (x+z+2xy)\vec{j} + (y+2zx)\vec{k}$$

and if $\phi(1,1,1) = 3$. find ϕ.

Solution

$$\nabla\phi = (y+y^2+z^2)\vec{i} + (x+z+2xy)\vec{j} + (y+2zx)\vec{k}$$

$$\frac{\partial\phi}{\partial x}\vec{i} + \frac{\partial\phi}{\partial y}\vec{j} + \frac{\partial\phi}{\partial z}\vec{k} = (y+y^2+z^2)\vec{i} + (x+z+2xy)\vec{j} + (y+2zx)\vec{k}$$

Equating the co-efficients of $\vec{i}, \vec{j}, \vec{k}$, we have

$$\frac{\partial\phi}{\partial x} = y+y^2+z^2 \longrightarrow ①$$

$$\frac{\partial\phi}{\partial y} = x+z+2xy \longrightarrow ②$$

$$\frac{\partial\phi}{\partial z} = y+2zx \longrightarrow ③$$

Integrating ① with respect to x, we get

$$\phi = yx + xy^2 + xz^2 + f_1(y,z)$$

Integrating ② w.r to y, we get

$$\phi = xy + zy + 2x\frac{y^2}{2} + f_2(x,z)$$

$$\phi = xy + zy + xy^2 + f_2(x,z)$$

Integrating ③ w.r to z, we get

$$\phi = yz + 2x\frac{z^2}{2} + f_3(x,y)$$

$$\phi = yz + xz^2 + f_3(x, y)$$

$$\therefore \phi(x, y, z) = xy + xy^2 + xz^2 + yz + c$$

Given $\phi(1, 1, 1) = 3$

$$\phi(1, 1, 1) = 1 + 1 + 1 + 1 + c$$

$$3 = 4 + c$$

$$c = 3 - 4$$

$$c = -1$$

$$\therefore \phi(x, y, z) = xy + xy^2 + xz^2 + yz - 1$$

Problem - 17

$\nabla\phi$ if $\nabla\phi = (6xy + z^3)\vec{i} + (3x^2 - z)\vec{j} + (3xz^2 - y)\vec{k}$
find ϕ.

Solution:

$$\nabla\phi = (6xy + z^3)\vec{i} + (3x^2 - z)\vec{j} + (3xz^2 - y)\vec{k}$$

$$\frac{\partial\phi}{\partial x}\vec{i} + \frac{\partial\phi}{\partial y}\vec{j} + \frac{\partial\phi}{\partial z}\vec{k} = (6xy - z^3)\vec{i} + (3x^2 - z)\vec{j} + (3xz^2 - y)\vec{k}$$

Equating the coefficients of $\vec{i}, \vec{j}, \vec{k}$, we have

$$\frac{\partial\phi}{\partial x} = 6xy + z^3 \longrightarrow ①$$

$$\frac{\partial\phi}{\partial y} = 3x^2 - z \longrightarrow ②$$

$$\frac{\partial\phi}{\partial z} = 3xz^2 - y \longrightarrow ③$$

Integrating ① with respect to x, we get

$$\phi = 6y\frac{x^2}{2} + xz^3 + f_1(y,z)$$

$$\phi = 3x^2y + xz^3 + f_1(y,z)$$

Integrating ② with respect to y, we get

$$\phi = 3x^2y - yz + f_2(x,z)$$

Integrating ③ with respecto to z, we get

$$\phi = 3x\frac{z^3}{3} - yz + f_3(x,y)$$

$$\phi = xz^3 - yz + f_3(x,y)$$

$$\therefore \phi(x,y,z) = 3x^2y + xz^3 - yz + c$$

Problem -18

Find ϕ if $\nabla\phi = (y+\sin z)\vec{i} + x\vec{j} + (x\cos z)\vec{k}$

Solution:

$$\nabla\phi = (y+\sin z)\vec{i} + x\vec{j} + (x\cos z)\vec{k}$$

$$\frac{\partial\phi}{\partial x}\vec{i} + \frac{\partial\phi}{\partial y}\vec{j} + \frac{\partial\phi}{\partial z}\vec{k} = (y+\sin z)\vec{i} + x\vec{j} + (x\cos z)\vec{k}$$

Equating the coefficients of $\vec{i}, \vec{j}, \vec{k}$, we have,

$$\frac{\partial\phi}{\partial x} = y+\sin z \longrightarrow ①$$

$$\frac{\partial\phi}{\partial y} = x \longrightarrow ②$$

$$\frac{\partial\phi}{\partial z} = x\cos z \rightarrow ③$$

Integrating ① with respect to x, we get,

$$\phi = xy + x\sin z + f_1(y, z)$$

Integrating ② with respect to y, we get,

$$\phi = xy + f_2(x, z)$$

Integrating ③ with respect to z, we get

$$\phi = x\sin z + f_3(x, y)$$

$$\therefore \phi(x, y, z) = xy + x\sin z + c$$

Note :

$$\nabla f(r) = f'(r) \cdot \hat{r}$$

Problem - 19

find ϕ. $\nabla\phi = 5r^3 \vec{r}$

Solution:

$$\nabla\phi = 5r^3 \vec{r}$$

$$\nabla\phi = 5r^3 \, r \cdot \hat{r}$$

$$\nabla\phi = 5r^4 \cdot \hat{r} \rightarrow ①$$

Put $\nabla\phi(r) = \phi'(r) \cdot \hat{r} \rightarrow ②$

from ① and ②

$$\phi'(r) \cdot \hat{r} = 5r^4 \cdot \hat{r}$$

$$\phi'(r) = 5r^4$$

$\int$ing, we get

$$\phi(r) = 5\frac{r^5}{5} + c$$

$$\phi(r) = r^5 + c$$

Problem - 20

If $\nabla\phi = (6r - 8r^2).\vec{r}$ and $\phi(2) = 4$.

find ϕ.

Solution:

$$\nabla\phi = (6r - 8r^2).\vec{r}$$

$$\nabla\phi = (6r - 8r^2)\, r.\hat{r}$$

$$\nabla\phi = (6r^2 - 8r^3).\hat{r} \longrightarrow ①$$

~~Put $\nabla\phi(r) = \phi\,\phi'(r).\hat{r}$~~

Put $\nabla\phi(r) = \phi'(r).\hat{r} \longrightarrow ②$

From ① and ②

$$\phi'(r).\hat{r} = (6r^2 - 8r^3).\hat{r}$$

$$\phi'(r) = (6r^2 - 8r^3)$$

$\int$ing, we get,

$$\phi(r) = \frac{6r^3}{3} - \frac{8r^4}{4} + c$$

$$\phi(r) = 2r^3 - 2r^4 + c$$

Given $\phi(2) = 4$

$$\phi(2) = 2(2)^3 - 2(2)^4 + c$$

$$4 = 16 - 32 + c$$

$c = 4 - 16 + 32$

$c = 20$

$\therefore \phi(r) = 2r^3 - 2r^4 + 20$

$\therefore \phi(r) = 2(r^3 - r^4 + 10)$

14/08/2020

One word.

Stat a unit vector normal to the surface $\phi = c$. Ans: $\frac{\nabla\phi}{|\nabla\phi|}$

Problem - 21

Find a unit vector normal to the surface $x^2 + y^2 + 2z^2 = 4$ at $(1, 1, 1)$

Solution:

$$\phi = x^2 + y^2 + 2z^2 - 4$$

$$\nabla\phi = \vec{i}(2x) + \vec{j}(2y) + \vec{k}(4z)$$

$$\nabla\phi(1,1,1) = 2\vec{i} + 2\vec{j} + 4\vec{k}$$

$$|\nabla\phi| = \sqrt{4 + 4 + 16} = \sqrt{24}$$

$$|\nabla\phi| = 2\sqrt{6}$$

Unit normal vector $\hat{n} = \frac{\nabla\phi}{|\nabla\phi|}$

$$= \frac{2\vec{i} + 2\vec{j} + 4\vec{k}}{2\sqrt{6}}$$

$$= \frac{2(\vec{i} + \vec{j} + 2\vec{k})}{2\sqrt{6}}$$

$$\hat{n} = \frac{\vec{i} + \vec{j} + 2\vec{k}}{\sqrt{6}}$$

Problem - 22

Find a unit vector normal to the surface $x^3 - xyz + y^3 = 1$ at $(1, 1, 1)$

Solution:

$$\phi = x^3 - xyz + y^3 - 1$$

$$\nabla\phi = \vec{i}\left(3x^2 - yz\right) + \vec{j}\left(-xz + 3y^2\right) + \vec{k}\left(-xy\right)$$

$$\nabla\phi = (3x^2 - yz)\vec{i} + (3y^2 - xz)\vec{j} - (xy)\vec{k}$$

$$\nabla\phi_{(1,1,1)} = (3-1)\vec{i} + (3-1)\vec{j} - \vec{k}$$

$$= 2\vec{i} + 2\vec{j} - \vec{k}$$

$$|\nabla\phi| = \sqrt{4+4+1} = \sqrt{9}$$

$$|\nabla\phi| = 3$$

Unit normal vector $\hat{n} = \dfrac{\nabla\phi}{|\nabla\phi|}$

$$\hat{n} = \frac{2\vec{i} + 2\vec{j} - \vec{k}}{3}$$

Problem - 23

Find the maximum value of the directional derivate of the function $\phi = 2x^2 + 3y^2 + 5z^2$ at the point $(1, 1, -4)$.

Solution:

$$\nabla\phi = (4x)\vec{i} + (6y)\vec{j} + (10z)\vec{k}$$

$$\nabla\phi(1,1,-4) = 4\vec{i} + 6\vec{j} - 40\vec{k}$$

$\therefore$ Maximum value of directional derivative $\Big\}$ $= |\nabla\phi|$

$$= \sqrt{16+36+1600}$$

$$= \sqrt{1652}$$

$$= 2\sqrt{413}$$

$2\,|\,1652$
$2\,|\,826$
413

$\therefore$ The maximum value of directional derivative is $2\sqrt{413}$

Problem - 24.

Find the direction in which $\phi = xy^2 + yz^2 + zx^2$ increases most rapidly at the point $(1, 2, -3)$.

Solution:

$$\nabla\phi = (y^2 + 2xz)\vec{i} + (2xy + z^2)\vec{j} + (2yz + x^2)\vec{k}$$

$$\nabla\phi(1,2,-3) = (4 + 2(-3))\vec{i} + (2(2) + 9)\vec{j} + (2(-6)+1)\vec{k}$$

$$= (4-6)\vec{i} + (4+9)\vec{j} + (-12+1)\vec{k}$$

$$= -2\vec{i} + 13\vec{j} - 11\vec{k}$$

$\therefore$ Direction of ϕ is increases most rapidly along $-2\vec{i} + 13\vec{j} - 11\vec{k}$

Problem - 25

Prove that the directional derivative $\phi = x^2y^2z$ at $(1, 2, 3)$ is a maximum along the direction $9\vec{i} + 3\vec{j} + \vec{k}$. Find this maximum directional derivative.

Solution:

$$\phi = x^3y^2z$$

$$\nabla\phi = (3x^2y^2z)\vec{i} + (2x^3yz)\vec{j} + (x^3y^2)\vec{k}$$

$$\nabla\phi(1,2,3) = (3(1)(4)(3))\vec{i} + (2(1)(2)(3))\vec{j} + ((1)(4))\vec{k}$$

$$= 36\vec{i} + 12\vec{j} + 4\vec{k} = 4(9\vec{i} + 3\vec{j} + \vec{k})$$

Maximum directional derivative is along $4(9\vec{i} + 3\vec{j} + \vec{k})$

$$\left.\begin{matrix}\text{Maximum directional} \\ \text{derivative}\end{matrix}\right\} = |\nabla\phi|$$

$$= 4\sqrt{81 + 9 + 1}$$

$$= 4\sqrt{91}$$

Maximum directional derivative is $4\sqrt{91}$

Problem - 26

The ~~tempetrat~~ temperature T at the point (x, y, z) in space is given by $T = xy^2z^3$. find the direction in which the rate of increase of temperature at (1, 1, 1) is the greatest.

Find this maximum rate.

Solution:

$$T = xy^2z^3$$

$$\nabla T = (y^2z^3)\vec{i} + (2xyz^3)\vec{j} + (3xy^2z^2)\vec{k}$$

$$\nabla T_{(1,1,1)} = \vec{i} + 2\vec{j} + 3\vec{k}$$

The temperature is greatest along $\vec{i} + 2\vec{j} + 3\vec{k}$

$$\left.\begin{array}{l}\text{Maximum directional}\\ \text{derivative}\end{array}\right\} = |\nabla T|$$

$$= \sqrt{1+4+9}$$

$$= \sqrt{14}.$$

Maximum directional derivative is $\sqrt{14}$.

Problem - 27

Find the directional derivative of $\phi = 3x^2 + 2y - 3z$ at the (1, 1, 1) in the direction specified by $2\vec{i} + 2\vec{j} - \vec{k}$ also find the maximum value of the directional derivative at ~~that~~ the point and the unit normal ~~vex~~ vector of the direction pertaining to this maximum.

Solution:

$$\phi = 3x^2 + 2y - 3z$$

$$\nabla\phi = (6x)\vec{i} + 2\vec{j} - 3\vec{k}$$

$$(\nabla\phi)_{(1,1,1)} = 6\vec{i} + 2\vec{j} - 3\vec{k}$$

$$\vec{e} = 2\vec{i} + 2\vec{j} - \vec{k}$$

$$\hat{e} = \frac{\vec{e}}{|\vec{e}|} = \frac{2\vec{i} + 2\vec{j} - \vec{k}}{\sqrt{4+4+1}} = \frac{2\vec{i} + 2\vec{j} - \vec{k}}{\sqrt{9}}$$

$$\hat{e} = \frac{1}{3}(2\vec{i} + 2\vec{j} - \vec{k})$$

Directional derivative of ϕ along $\vec{e}$ } $= \nabla\phi \cdot \hat{e}$

$= (6\vec{i} + 2\vec{j} - 3\vec{k}) \cdot \frac{1}{3}(2\vec{i} + 2\vec{j} - \vec{k})$

$= \frac{1}{3}[12 + 4 + 3]$

$= \frac{1}{3}[19]$

Directional derivative of ϕ along $\vec{e}$ } $= \frac{19}{3}$

Maximum value of directional derivative } $= |\nabla\phi|$

$= \sqrt{36 + 4 + 9}$

$= \sqrt{49}$

Maximum value of directional derivative } $= 7$

Unit normal vector $= \frac{\nabla\phi}{|\nabla\phi|}$

Unit normal vector $= \frac{6\vec{i} + 2\vec{j} - 3\vec{k}}{7}$

19/8/2020 Problem - 28

Find the angle between surfaces $x^2 + yz = 2$ and $x + 2y - z = 2$ at $(1,1,1)$

Solution:

$$\phi_1 = x^2 + yz - 2$$

$$\nabla\phi_1 = (2x)\vec{i} + z\vec{j} + y\vec{k}$$

$$\nabla\phi_1(1,1,1) = 2\vec{i} + \vec{j} + \vec{k}$$

$$\phi_2 = x + 2y - z - 2$$

$$\nabla\phi_2 = \vec{i} + 2\vec{j} - \vec{k}$$

$$(\nabla\phi_2)_{(1,1,1)} = \vec{i} + 2\vec{j} - \vec{k}$$

$$\cos\theta = \frac{\nabla\phi_1 \cdot \nabla\phi_2}{|\nabla\phi_1|\,|\nabla\phi_2|}$$

$$= \frac{(2\vec{i} + \vec{j} + \vec{k}) \cdot (\vec{i} + 2\vec{j} - \vec{k})}{\sqrt{4+1+1}\ \sqrt{1+4+1}}$$

$$= \frac{2 + 2 - 1}{\sqrt{6}\sqrt{6}}$$

$$= \frac{3}{6}$$

$$\cos\theta = \frac{1}{2}$$

$$\cos\theta = \cos\frac{\pi}{3}$$

$$\theta = \frac{\pi}{3}$$

Problem- 29

Find the angle between the surfaces $x^2 - y^2 - z^2 - 11 = 0$ and $xy + yz - zx - 18 = 0$ at ~~6,~~ $(6, 4, 3)$.

Solution:

$$\phi_1 = x^2 - y^2 - z^2 - \cancel{11}$$

$$\nabla\phi_1 = (2x)\vec{i} - (2y)\vec{j} - (2z)\vec{k}$$

~~$\nabla\phi_1(6,4,3) = 12\vec{i} - 8y$~~

$$\nabla\phi_1(6,4,3) = 12\vec{i} - 8\vec{j} - 6\vec{k}$$

$$\phi_2 = xy + yz - zx - 18$$

$$\nabla\phi_2 = (y-z)\vec{i} + (x+z)\vec{j} + (y-x)\vec{k}$$

$$\nabla\phi_2(6,4,3) = (4-3)\vec{i} + (6+3)\vec{j} + (4-6)\vec{k}$$

$$= \vec{i} + 9\vec{j} - 2\vec{k}$$

$2\,|\,244$
$2\,|\,122$
61

$2\,|\,86$
43

~~$\cos\phi$~~ $$= \frac{\nabla\phi_1 \cdot \nabla\phi_2}{|\nabla\phi_1|\,|\nabla\phi_2|}$$

$$= \frac{(12\vec{i} - 8\vec{j} - 6\vec{k})\cdot(\vec{i} + 9\vec{j} - 2\vec{k})}{\sqrt{144 + 64 + 36}\,\sqrt{1 + 81 + 4}}$$

$$= \frac{12 - 72 + 12}{\sqrt{244}\,\sqrt{86}} = \frac{-48}{2\sqrt{61}\,\sqrt{86}}$$

$$\cos\phi = \frac{-24}{\sqrt{61}\,\sqrt{86}}$$

Problem - 30

Find the angle between the normals to the surface $xy - z^2 = 0$ at the points $(1, 4, -2)$ and $(-3, -3, 3)$

Solution:

$$\phi = xy - z^2$$

$$(\nabla\phi) = y\vec{i} + x\vec{j} - 2z\vec{k}$$

$$(\nabla\phi_1)_{(1,4,-2)} = 4\vec{i} + \vec{j} + 4\vec{k}$$

~~$(\nabla\phi_2)_{(-3,3,}$~~

$$(\nabla\phi_2)_{(-3,-3,3)} = -3\vec{i} - 3\vec{j} - 6\vec{k}$$

$$\cos\phi = \frac{\nabla\phi_1 \cdot \nabla\phi_2}{|\nabla\phi_1|\,|\nabla\phi_2)}$$

$$= \frac{(4\vec{i} + \vec{j} + 4\vec{k}).(-3\vec{i} - 3\vec{j} - 6\vec{k})}{\sqrt{16+1+16}\ \sqrt{9+9+36}}$$

2 | 54
3 | 27
3 | 9
3

$$= \frac{-12 - 3 - 24}{\sqrt{33}\ \sqrt{54}}$$

$$= \frac{-39}{\sqrt{33}\ 3\sqrt{6}}$$

2 | 198
3 | 99
3 | 33
11

$$\cos\phi = \frac{-13}{\sqrt{33}\sqrt{6}}$$

$$\cos\phi = \frac{-13}{\sqrt{198}} = \frac{-13}{3\sqrt{22}}$$

Problem - 31

Find the equation of the tangent normal to the surface $x^2+2y^2+3z^2=6$ at $(1,-1,1)$

Solution:

$$\phi = x^2+2y^2+3z^2-6$$

$$\nabla\phi = 2x\vec{i}+4y\vec{j}+6z\vec{k}$$

$$(\nabla\phi)_{(1,-1,1)} = 2\vec{i}-4\vec{j}+6\vec{k}$$

$$\therefore \alpha \neq \quad \therefore (a, b, c) = (2, -4, 6)$$

$$(x_1, y_1, z_1) = (1, -1, 1)$$

$\therefore$ Equation of the tangent plane is

$$a(x-x_1)+b(y-y_1)+c(z-z_1)=0$$

$$2(x-1)+(-4)(y+1)+6(z-1)=0$$

$$2x-2-4y-4+6z-6=0$$

$$2x-4y+6z-12=0$$

$(\div 2)$

$$x-2y+3z-6=0$$

Problem - 32

Find the equation of the tangent normal to the surface $x^2-4y^2+3z^2+4=0$ at $(3,2,1)$

Solution

$$\phi = x^2-4y^2+3z^2+4$$

$$\nabla\phi = 2x\vec{i}-8y\vec{j}+6z\vec{k}$$

$$(\nabla\phi)_{(3,2,1)} = 6\vec{i} - 16\vec{j} + 6\vec{k}$$

$$(a, b, c) = (6, -16, 6)$$

$$(x_1, y_1, z_1) = (3, 2, 1)$$

$\therefore$ The equation of the tangent plane is

$$a(x-x_1) + b(y-y_1) + c(z-z_1) = 0$$

$$6(x-3) + (-16)(y-2) + 6(z-1) = 0$$

$$6x - 18 - 16y + 32 + 6z - 6 = 0$$

$$6x - 16y + 6z + 8 = 0$$

$(\div 2)$

$$3x - 8y + 3z + 4 = 0$$

20/08/2020 Note:

$$\nabla = \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z} = \Sigma\vec{i}\frac{\partial}{\partial x}$$

$$\nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z} = \Sigma\vec{i}\frac{\partial\phi}{\partial x}$$

Theorem:

If ϕ and ψ are scalar point function,

i) $\nabla(k\phi) = k(\nabla\phi)$ where k is a scalar.

ii) $\nabla(\phi+\psi) = \nabla\phi + \nabla\psi$

iii) $\nabla(\phi\psi) = (\nabla\phi)\psi + (\nabla\psi)\phi$

iv) $\nabla\left(\frac{\phi}{\psi}\right) = \frac{\psi(\nabla\phi) - \phi(\nabla\psi)}{\psi^2}$

Proof:

~~i) $\nabla(k\phi) =$~~

i) $\nabla(k\phi) = \sum \vec{i}\,\frac{\partial}{\partial x}(k\phi)$

$$= k\sum \vec{i}\,\frac{\partial\phi}{\partial x}$$

$$\nabla(k\phi) = k(\nabla\phi)$$

ii) $\nabla(\phi+\psi) = \sum \vec{i}\,\frac{\partial}{\partial x}(\phi+\psi)$

$$= \sum \vec{i}\left(\frac{\partial\phi}{\partial x} + \frac{\partial\psi}{\partial x}\right)$$

$$= \sum \vec{i}\,\frac{\partial\phi}{\partial x} + \sum \vec{i}\,\frac{\partial\psi}{\partial x}$$

$$\nabla(\phi+\psi) = \nabla\phi + \nabla\psi$$

iii) $\nabla(\phi\psi) = \sum \vec{i}\,\frac{\partial}{\partial x}(\phi\psi)$

$$= \sum \vec{i}\left[\phi\frac{\partial\psi}{\partial x} + \psi\frac{\partial\phi}{\partial x}\right].$$

$$= \phi\sum \vec{i}\,\frac{\partial\psi}{\partial x} + \cancel{\psi\nabla}\ \psi\sum \vec{i}\,\frac{\partial\phi}{\partial x}$$

$$\nabla(\phi\psi) = \phi(\nabla\psi) + \psi(\nabla\phi)$$

$$\therefore\ \nabla(\phi\psi) = (\nabla\psi)\,\phi + \psi\,(\nabla\phi)$$

iv). $\nabla\left(\frac{\phi}{\psi}\right) = \sum \vec{i}\,\frac{\partial}{\partial x}\left(\frac{\phi}{\psi}\right)$

$$= \sum \vec{i}\left[\frac{\psi\frac{\partial\phi}{\partial x} - \phi\frac{\partial\psi}{\partial x}}{\psi^2}\right]$$

$$= \frac{\psi \sum \vec{i} \frac{\partial \phi}{\partial x} - \phi \sum \vec{i} \frac{\partial \psi}{\partial x}}{\psi^2}$$

$$\nabla\left(\frac{\phi}{\psi}\right) = \frac{\psi(\nabla \phi) - \phi(\nabla \psi)}{\psi^2}$$

Problem - 33

If $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ and $|\vec{r}| = r$,

prove that.

So i) $\nabla f(r) = f'(r)\,\hat{r}$

ii) $\nabla\left(\frac{1}{r}\right) = \frac{-\hat{r}}{r^2}$ (or) $\frac{-\vec{r}}{r^3}$

ⓧ iii) $\nabla(r^n) = n r^{n-1}\hat{r}$ (or) $n r^{n-2}\vec{r}$

iv) $\nabla(\log r) = \frac{\hat{r}}{r}$ (or) $\frac{\vec{r}}{r^2}$

Solution:

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$$

$$|\vec{r}| = \sqrt{x^2 + y^2 + z^2}$$

$$r^2 = x^2 + y^2 + z^2 \longrightarrow ①$$

Differentiate ① with respect to x, we have

$$2r \frac{\partial r}{\partial x} = 2x$$

$$\frac{\partial r}{\partial x} = \frac{x}{r}$$

|||ly $\frac{\partial r}{\partial y} = \frac{y}{r}$, $\frac{\partial r}{\partial z} = \frac{z}{r}$.

i) $\nabla f(r) = \sum \vec{i}\, \frac{\partial}{\partial x}\left(f(r)\right)$

$$= \sum \vec{i}\; f'(r)\; \frac{\partial r}{\partial x}$$

$$= f'(r) \sum \vec{i}\; \frac{\partial r}{\partial x} = f'(r) \sum \vec{i}\left(\frac{x}{r}\right)$$

$$= f'(r)\left[\frac{x\vec{i} + y\vec{j} + z\vec{k}}{r}\right]$$

$$= f'(r)\left[\frac{\vec{r}}{r}\right]$$

$$= f'(r)\cancel{\left[\frac{r \cdot \vec{r}}{r}\right]}\left[\frac{r \cdot \hat{r}}{r}\right]$$

$$\nabla f(r) = f'(r) \cdot \hat{r}$$

ii) $\nabla\left(\frac{1}{r}\right) = \sum \vec{i}\; \frac{\partial}{\partial x}\left(\frac{1}{r}\right)$

$$= \sum \vec{i}\left(\frac{-1}{r^2}\right)\frac{\partial r}{\partial x}$$

$$= \frac{-1}{r^2} \sum i\, \frac{\partial r}{\partial x} = \frac{-1}{r^2} \sum \vec{i}\left(\frac{x}{r}\right)$$

$$= \frac{-1}{r^2}\left[\frac{x\vec{i} + y\vec{j} + z\vec{k}}{r}\right]$$

$$= \frac{-\left[\vec{r}\right]}{r^3}$$

$$= \frac{-r \cdot \hat{r}}{r^3}$$

$$\nabla \frac{1}{r} = \frac{-\hat{r}}{r^2}$$

iii) $\nabla(r^n) = \Sigma \vec{i} \frac{\partial}{\partial x}(r^n)$

$= \Sigma \vec{i}\, n r^{n-1} \frac{\partial r}{\partial x}$

$= n r^{n-1} \Sigma \vec{i} \frac{\partial r}{\partial x} = n r^{n-1} \Sigma \vec{i} \frac{x}{r}$

$= n r^{n-1} \left[\frac{x\vec{i} + y\vec{j} + z\vec{k}}{r}\right]$

$= n r^{n-1-1}\, \vec{r}$

$= n r^{n-2}\, r.\hat{r}$

$\nabla(r^n) = n r^{n-1} \hat{r}$

iv) $\nabla(\log r) = \Sigma \vec{i} \frac{\partial}{\partial x}(\log r)$

$= \Sigma \vec{i} \frac{1}{r} \frac{\partial r}{\partial x}$

$= \frac{1}{r} \Sigma \vec{i} \frac{\partial r}{\partial x} = \frac{1}{r} \Sigma \vec{i}\left(\frac{x}{r}\right)$

$= \frac{1}{r}\left[\frac{x\vec{i} + y\vec{j} + z\vec{k}}{r}\right]$

$= \frac{\vec{r}}{r^2}$

$= \frac{r.\hat{r}}{r^2}$

$\nabla(\log r) = \frac{\hat{r}}{r}$

21/08/2020

Unit - II

Divergence:

If $\vec{v} = v_1\vec{i} + v_2\vec{j} + v_3\vec{k}$ is a vector point function, then the scalar function (a constant) $\frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}$ is called divergence of $\vec{v}$ is denoted by div $\vec{v}$.

Note:

$$\nabla.\vec{v} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right).\left(v_1\vec{i} + v_2\vec{j} + v_3\vec{k}\right)$$

$$\nabla.\vec{v} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}$$

$$\nabla.\vec{v} = \text{div}\,\vec{v}$$

$$\therefore \text{div}\,\vec{v} = \nabla.\vec{v}$$

Curl:

If $\vec{v} = v_1\vec{i} + v_2\vec{j} + v_3\vec{k}$ is a vector point function, then the vector function

$$\vec{i}\left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right) + \vec{j}\left(\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x}\right) + \vec{k}\left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right)$$

is called curl or ~~read~~ relation of $\vec{v}$ and is denoted by curl $\vec{v}$.

Note:

$$\nabla \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix}$$

$$= \vec{i}\left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right) - \vec{j}\left(\frac{\partial v_3}{\partial x} - \frac{\partial v_1}{\partial z}\right) + \vec{k}\left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right)$$

$$= \vec{i}\left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right) + \vec{j}\left(\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x}\right) + \vec{k}\left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right)$$

= curl

Note:

If $\vec{a}$ is a constant vector then $\nabla . \vec{a} = 0$

and $\nabla \times \vec{a} = 0$

Definition:

If $\nabla . \vec{v} = 0$, then $\vec{v}$ is said to be ~~scalar potential vector. solenodial~~ solenoidal vector.

If $\nabla \times \vec{v} = 0$, then $\vec{v}$ is said to be irrotational vector.

Note:

$$\nabla \times \nabla \phi = 0$$

Scalar Potential:

Given a vector point function $\vec{F}$. If there exists a scalar point function ϕ, such ~~there~~ that $\vec{F} = \nabla\phi$. Then ϕ is called the scalar potential of $\vec{F}$.

In such case $\nabla \times \vec{F} = \nabla \times \nabla\phi = 0$.

~~$\therefore$ F is irrotatio~~

$\therefore$ F is irrotational.

Conversely, if $\vec{F}$ is irrotational, ~~if~~ it can be proved that, there exists a scalar potential ϕ, such that

$$\vec{F} = \nabla\phi$$

Problem - 1

Find $\nabla . \vec{r}$ and $\nabla \times \vec{r}$ if $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

Solution:

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$$

$$\nabla = \frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}$$

$$\nabla . \vec{r} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z}$$

$$= 1 + 1 + 1$$

$$\nabla . \vec{r} = 3$$

∇

$$\nabla \times \vec{r} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix}$$

$$= \vec{i}(0-0) - \vec{j}(0-0) + \vec{k}(0-0)$$

$$\nabla \times \vec{r} = \vec{0}$$

Problem - 2

Find the divergence and curl of ~~$x\vec{i}+y\vec{j}+z\vec{k}$~~ $x^2\vec{i}+y^2\vec{j}+z^2\vec{k}$

Solution:

$$\vec{r} = x^2\vec{i} + y^2\vec{j} + z^2\vec{k}$$

$$\nabla = \frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}$$

Divergence

$$\nabla \cdot \vec{r} = \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial y}(y^2) + \frac{\partial}{\partial z}(z^2)$$

$$= 2x + 2y + 2z$$

$$\nabla \cdot \vec{r} = 2(x+y+z)$$

Curl:

$$\nabla \times \vec{r} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & y^2 & z^2 \end{vmatrix}$$

$$= \vec{i}(0-0) - \vec{j}(0-0) + \vec{k}(0-0)$$

$$\text{Curl} = \vec{0}$$

Problem - 3

Find $\nabla\cdot\vec{F}$ and $\nabla\times\vec{F}$ at the point $(1,-1,1)$ if $\vec{F} = xz^3\vec{i} - 2x^2yz\vec{j} + 2yz^4\vec{k}$

Solution:

$$\vec{F} = xz^3\vec{i} - 2x^2yz\vec{j} + 2yz^4\vec{k}$$

$$\nabla = \frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}$$

$$\nabla\cdot\vec{F} = \frac{\partial}{\partial x}(xz^3) + \frac{\partial}{\partial y}(-2x^2yz) + \frac{\partial}{\partial z}(2yz^4)$$

$$\nabla\cdot\vec{F} = z^3 - 2x^2z + 4yz^3$$

$$(\nabla\cdot\vec{F})_{(1,-1,1)} = (1)^3 - 2(1)^2(1) + 8(-1)(1)^3$$

$$= 1 - 2 - 8$$

$$(\nabla\cdot\vec{F})_{(1,-1,1)} = -9$$

$$\nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz^3 & -2x^2yz & 2yz^4 \end{vmatrix}$$

$$= \vec{i}\left[\frac{\partial}{\partial y}(2yz^4) - \frac{\partial}{\partial z}(-2x^2yz)\right] - \vec{j}\left[\frac{\partial}{\partial x}(2yz^4) - \frac{\partial}{\partial z}(xz^3)\right] + \vec{k}\left[\frac{\partial}{\partial x}(-2x^2yz) - \frac{\partial}{\partial y}(xz^3)\right]$$

$$= \vec{i}(2z^4 + 2x^2y) - \vec{j}(0 - 3xz^2) + \vec{k}(-4xyz - 0)$$

$$\nabla \times \vec{F} = (2z^4 + 2x^2 y)\vec{i} + (3xz^2)\vec{j} - (4xyz)\vec{k}$$

$$(\nabla \times \vec{F})_{(1,-1,1)} = (2(1) + 2(1)(-1))\vec{i} + (3(1)(1))\vec{j} - (4(1)(-1)(1))\vec{k}$$

$$= (2-2)\vec{i} + 3\vec{j} + 4\vec{k}$$

$$(\nabla \times \vec{F})_{(1,-1,1)} = 3\vec{j} + 4\vec{k}$$

Problem - 4

Show that ~~$\vec{A} = x^2 y^2 \vec{i}$~~

$\vec{A} = x^2 z^2 \vec{i} + xyz^2 \vec{j} - xz^3 \vec{k}$ is solenoidal.

Solution:

~~$\nabla \cdot \vec{A} = \frac{\partial}{\partial x}(2xz^2)$~~

$$\vec{A} = x^2 z^2 \vec{i} + xyz^2 \vec{j} - xz^3 \vec{k}$$

$$\nabla = \frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}$$

$$\nabla \cdot \vec{A} = \frac{\partial}{\partial x}(x^2 z^2) + \frac{\partial}{\partial y}(xyz^2) + \frac{\partial}{\partial z}(-xz^3)$$

$$= 2xz^2 + xz^2 - 3xz^2$$

$$\nabla \cdot \vec{A} = 3xz^2 - 3xz^2$$

$$\nabla \cdot \vec{A} = 0$$

$\vec{A}$ is a solenoidal

Problem - 5

Define solenoidal vector. Show that

$3x^2 y \vec{i} - 4xy^2 \vec{j} + 2xyz \vec{k}$ is solenoidal.

Solution:

$$\vec{A} = 3x^2y\,\vec{i} - 4xy^2\,\vec{j} + 2xyz\,\vec{k}$$

$$\nabla = \frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}$$

$$\nabla\cdot\vec{A} = \frac{\partial}{\partial x}(3x^2y) + \frac{\partial}{\partial y}(-4xy^2) + \frac{\partial}{\partial z}(2xyz)$$

$$= 6xy - 8xy + 2xy$$

$$= 8xy - 8xy$$

$$\nabla\cdot\vec{A} = 0$$

$\therefore \vec{A}$ is a solenoidal.

Problem - 6

Find the value of m if the vector

$(x+2y)\,\vec{i} + (my+4z)\,\vec{j} + (5z+6x)\,\vec{k}$ is solenoidal vector.

Solution:

$$\vec{F} = (x+2y)\vec{i} + (my+4z)\,\vec{j} + (5z+6x)\vec{k}$$

Given : $\vec{F}$ is solenoidal vector.

$\therefore\ \nabla\cdot\vec{F} = 0$

$$\frac{\partial}{\partial x}(x+2y) + \frac{\partial}{\partial y}(my+4z) + \frac{\partial}{\partial z}(5z+6x) = 0$$

$$1 + m + 5 = 0$$

$$m + 6 = 0$$

$$m = -6$$

Problem - 7

Determine the constant 'a' so that the vector

$\vec{F} = (x+2y)\vec{i} + (y-2z)\vec{j} + (x+az)\,\vec{k}$ is solenoidal.

Solution:

$$\vec{F} = (x+2y)\vec{i} + (y-2z)\vec{j} + (x+az)\vec{k}$$

Given $\vec{F}$ is solenoidal vector.

$$\therefore \nabla \cdot \vec{F} = 0$$

$$\frac{\partial}{\partial x}(x+2y) + \frac{\partial}{\partial y}(y-2z) + \frac{\partial}{\partial z}(x+az) = 0$$

$$1+1+a = 0$$

$$2+a = 0$$

$$a = -2$$

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Problem - 8

Find curl $\vec{F}$, where $\vec{F} = x^2z\vec{i} - 2yz^2\vec{j} + xy^2z\vec{k}$ at $(1,-1,1)$.

Solution:

$$\text{curl } \vec{F} = \nabla \times \vec{F}$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2z & -2yz^2 & xy^2z \end{vmatrix}$$

$$= \vec{i}\left[\frac{\partial}{\partial y}(xy^2z) + \frac{\partial}{\partial z}(2yz^2)\right] - \vec{j}\left[\frac{\partial}{\partial x}(xy^2z) - \frac{\partial}{\partial z}(x^2z)\right] + \vec{k}\left[\frac{\partial}{\partial x}(-2yz^2) - \frac{\partial}{\partial y}(x^2z)\right]$$

$$= \vec{i}\left[2xyz + 4yz\right] - \vec{j}\left[y^2z - x^2\right] + \vec{k}\left[0-0\right]$$

$$\nabla \cdot \vec{F} = (2xyz + 4yz)\vec{i} - [y^2z - x^2]\vec{j}$$

At $(1, -1, 1)$

$$\text{Curl } \vec{F} = [2(1)(-1)(1) + 4(-1)(1)]\vec{i} - [(-1)^2(1) - (1)^2]\vec{j}$$

$$= (-2-4)\vec{i} - (1-1)\vec{j}$$

$$\vec{F} = -6\vec{i}$$

Problem - 9

Show that $\vec{F} = yz\,\vec{i} + zx\,\vec{j} + xy\,\vec{k}$ is irrotational.

Solution :

$$\vec{F} = yz\,\vec{i} + zx\,\vec{j} + xy\,\vec{k}$$

$$\nabla = \frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}$$

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & zx & xy \end{vmatrix}$$

$$= \vec{i}\left[\frac{\partial}{\partial y}(xy) - \frac{\partial}{\partial z}(zx)\right] - \vec{j}\left[\frac{\partial}{\partial x}(xy) - \frac{\partial}{\partial z}(yz)\right] + \vec{k}\left[\frac{\partial}{\partial x}(zx) - \frac{\partial}{\partial y}(yz)\right]$$

$$= \vec{i}[x - x] - \vec{j}[y - y] + \vec{k}[z - z]$$

$$= \vec{i}(0) - \vec{j}(0) + \vec{k}(0)$$

$$\nabla \times \vec{F} = \vec{0}$$

$\therefore$ $\vec{F}$ is irrotational.

Problem - 10

Find the value of a if $\vec{F} = axy\vec{i} + (x^2 + 2yz)\vec{j} + y^2\vec{k}$ is irrotational.

Solution:

$\vec{F}$ is irrotational

$$\nabla \times \vec{F} = 0$$

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ axy & x^2+2yz & y^2 \end{vmatrix} = 0$$

$$\vec{i}\left(2y - 2y\right) - \vec{j}\left(0 - 0\right) + \vec{k}\left(2x \cancel{+} -ax\right) = 0$$

$$\vec{i}(0) - \vec{j}(0) + \vec{k}(2x - ax) = 0$$

$$\vec{k}(2x - ax) = 0$$

$$2x - ax = 0$$

$$ax = 2x$$

$$a = 2$$

Problem - 11

Find the value of the constants a, b, c. So that $\vec{F} = (x+2y+az)\vec{i} + (bx-3y-z)\vec{j} + (4x+cy+2z)\vec{k}$ is irrotational

Solution:

$\vec{F}$ is irrotational

$$\therefore \nabla \times \vec{F} = 0$$

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+2y+az & bx-3y-z & 4x+cy+2z \end{vmatrix} = 0$$

$$\vec{i}\left[\frac{\partial}{\partial y}(4x+cy+2z) - \frac{\partial}{\partial z}(bx-3y-z)\right] -$$

$$\vec{j}\left[\frac{\partial}{\partial x}(4x+cy+2z) - \frac{\partial}{\partial z}(x+2y+az)\right] +$$

$$\vec{k}\left[\frac{\partial}{\partial x}(bx-3y-z) - \frac{\partial}{\partial y}(x+2y+az)\right] = 0$$

$$\vec{i}\left[c-(-1)\right] - \vec{j}\left[4-a\right] + \vec{k}\left[b-2\right] = 0$$

$$\therefore c+1=0, \quad -4+a=0, \quad b-2=0$$

$$c=-1, \quad a=4, \quad b=2$$

$$\therefore a=4, \; b=2, \; c=-1$$

Problem-12

Show that $\vec{F} = (y^2 \cos x + z^3)\vec{i} + (2y \sin x - 4)\vec{j} + 3xz^2\vec{k}$ is irrotational and find to scalar potential.

Solution:

$$\nabla \times \vec{F} = 0$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2\cos x+z^3 & 2y\sin x-4 & 3xz^2 \end{vmatrix}$$

$$= \vec{i}\left[\frac{\partial}{\partial y}(3xz^2) - \frac{\partial}{\partial z}(2y\sin x - 4)\right] -$$

$$\vec{j}\left[\frac{\partial}{\partial x}(3xz^2) - \frac{\partial}{\partial z}(y^2\cos x + z^3)\right] +$$

$$\vec{k}\left[\frac{\partial}{\partial x}(2y\sin x - 4) - \frac{\partial}{\partial y}(y^2\cos x - z^3)\right]$$

$$= \vec{i}(0-0) - \vec{j}(3z^2 - 3z^2) + \vec{k}(2y\cos x - 2y\cos x) = 0$$

$$= \vec{i}(0) - \vec{j}(0) + \vec{k}$$

$$= \vec{0}$$

$$\therefore \nabla \times \vec{F} = \vec{0}$$

$\therefore \vec{F}$ is irrotational.

Let ϕ be the scalar potential of $\vec{F}$.

$$\therefore \vec{F} = \nabla\phi$$

$$\vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z} = (y^2\cos x + z^3)\vec{i} + (2y\sin x - 4)\vec{j} + 3xz^2\vec{k}$$

$$\therefore \frac{\partial\phi}{\partial x} = y^2\cos x + z^3 \longrightarrow ①$$

$$\frac{\partial\phi}{\partial y} = 2y\sin x - 4 \longrightarrow ②$$

$$\frac{\partial\phi}{\partial z} = 3xz^2 \longrightarrow ③$$

Integrate ① with respect to x, we get,

$$\int d\phi = \int (y^2\cos x + z^3)\, dx$$

$$\phi = y^2\sin x + xz^3 + f(y,z) \rightarrow ④$$

Integrate ② with respect to y, we get,

$$\int \partial\phi = \int (2y\sin x - 4)\, \partial y$$

$$\phi = \frac{2y^2\sin x}{2} - 4y + g(z,x)$$

$$\phi = y^2\sin x - 4y \longrightarrow ⑤$$

Integrate ③ with respect to z, we get,

$$\int \partial\phi = \int 3xz^2\, dz$$

$$\phi = \frac{3xz^3}{3} + h(x,y)$$

$$\phi = xz^3 + h(x,y) \longrightarrow ⑥$$

From ④, ⑤, ⑥

$$\phi(x,y,z) = y^2\sin x + xz^3 - 4y + c$$

Problem-13

Show that $\vec{F} = (6xy + z^3)\vec{i} + (3x^2 - z)\vec{j} + (3xz^2 - y)\vec{k}$ is irrotational and find its scalar potential.

Solution:

T.P $\nabla \times \vec{F} = 0$

$$\therefore \nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cancel{6xy - z^3} & \cancel{3} & \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 6xy + z^3 & 3x^2 - z & 3xz^2 - y \end{vmatrix}$$

$$= \vec{i}\Big[\frac{\partial}{\partial y}(3xz^2-y) - \frac{\partial}{\partial z}(3x^2-z)\Big] - \vec{j}\Big[\frac{\partial}{\partial x}(3xz^2-y)$$

$$- \frac{\partial}{\partial z}(6xy+z^3))\Big] + \vec{k}\Big[\frac{\partial}{\partial x}(3x^2-z) - \frac{\partial}{\partial y}(6xy+z^3)\Big]$$

$$= \vec{i}\big[-1-(-1)\big] - \vec{j}\big[3z^2 - 3z^2\big] + \vec{k}\big[6x - 6x\big]$$

$$= \vec{i}(0) - \vec{j}(0) + \vec{k}(0)$$

$$= \vec{0}$$

$\therefore$ $\vec{F}$ is irrotational

Let ϕ be the scalar potential of $\vec{F}$.

$$\therefore \vec{F} = \nabla\phi$$

$$\vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z} = (6xy+z^3)\vec{i} + (3x^2-z)\vec{j} + (3xz^2-y)\vec{k}$$

$$\therefore \frac{\partial\phi}{\partial x} = 6xy + z^3$$

$$\partial\phi = (6xy+z^3)\,dx \longrightarrow ①$$

$$\frac{\partial\phi}{\partial y} = 3x^2 - z$$

$$\partial\phi = (3x^2-z)\,dy \longrightarrow ②$$

$$\frac{\partial\phi}{\partial z} = (3xz^2-y)$$

$$\partial\phi = (3xz^2-y)\,dz \longrightarrow ③$$

Integrating ① with respect to x, we get

$$\cancel{\int\partial\phi = \int(3x^2-z)\,dy}$$

$$\phi \cancel{=}$$

$$\int \partial\phi = \int (6xy + z^3)\,\partial x$$

$$\phi = \frac{6x^2y}{2} + xz^3$$

$$\phi = 3x^2y + xz^3 + \underset{f(y,z)}{\longrightarrow} ④$$

Integrating ② with respect to y, we get

$$\int \partial\phi = \int (3x^2 - z)\,\partial y$$

$$\phi = 3x^2y - yz + g(z,x) \longrightarrow ⑤$$

Integrating ③ with respect to z, we get,

$$\int \partial\phi = \int (3xz^2 - y)\,\partial z$$

$$\phi = \frac{3xz^3}{3} - yz + h(x,y)$$

$$\phi = xz^3 - yz + h(x,y) \longrightarrow ⑥$$

From ④, ⑤, ⑥

$$\phi(x,y,z) = 3x^2y + xz^3 - yz + c$$

Problem-14

Find the value of m if

$\vec{F} = (6xy + z^3)\vec{i} + (mx^2 - z)\vec{j} + (3xz^2 - y)\vec{k}$ is

irrotational and for this value of m find ϕ such that $\vec{F} = \nabla\phi$.

Solution:

$\vec{F}$ is irrotational.

$\therefore \nabla \times \vec{F} = 0$

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 6xy+z^3 & mx^2-z & 3xz^2-y \end{vmatrix} = 0$$

$$\vec{i}\left[\frac{\partial}{\partial y}(3xz^2-y) - \frac{\partial}{\partial z}(mx^2-z)\right] -$$

$$\vec{j}\left[\frac{\partial}{\partial x}(3xz^2-y) - \frac{\partial}{\partial z}(6xy+z^3)\right] +$$

$$\vec{k}\left[\frac{\partial}{\partial x}(mx^2-z) - \frac{\partial}{\partial y}(6xy+z^3)\right] = 0$$

$$\vec{i}\left[-1-(-1)\right] - \vec{j}\left[3z^2-3z^2\right] + \vec{k}\left[2mx-6x\right] = 0$$

$$\vec{i}\,(0) - \vec{j}\,(0) + \vec{k}\,x(2m-6) = 0$$

$$\vec{k}\,x\,(2m-6) = 0$$

$$2m-6 = 0$$

$$2m = 6$$

$$m = 3$$

By previous item

$$\vec{F} = (6xy+z^3)\vec{i}\ (3x^2-z)\vec{j} + (3xz^2-y)\vec{k}$$

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Problem - 15

$\vec{F} = xyz\,\vec{i} + xyz^2\,\vec{j} + x^2yz\,\vec{k}$. Find ~~div curl~~ div curl $\vec{F}$ (or) $\nabla\cdot(\nabla\times\vec{F})$.

Solution:

$$\text{Curl } \vec{F} = \nabla\times\vec{F}$$

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xyz & xyz^2 & x^2yz \end{vmatrix}$$

$$= \vec{i}\left[\frac{\partial}{\partial y}(x^2yz) - \frac{\partial}{\partial z}(xyz^2)\right] - \vec{j}\left[\frac{\partial}{\partial x}(x^2yz)\right.$$

$$\left. - \frac{\partial}{\partial z}(xyz)\right] + \vec{k}\left[\frac{\partial}{\partial x}(xyz^2) - \frac{\partial}{\partial y}(xyz)\right]$$

~~$= \vec{i}[x^2 z$~~

$$\nabla \times \vec{F} = \vec{i}[x^2z - 2xyz] - \vec{j}[2xyz - xy] + \vec{k}[yz^2 - xz]$$

$$\text{div curl } \vec{F} = \nabla \cdot (\nabla \times \vec{F})$$

$$= \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}\right) \cdot$$

$$(x^2z - 2xyz)\vec{i} - (2xyz - xy)\vec{j} + (yz^2 - xz)\vec{k}$$

$$= \frac{\partial}{\partial x}(x^2z - 2xyz) + \frac{\partial}{\partial y}(xy - 2xyz) +$$

$$\frac{\partial}{\partial z}(yz^2 - xz)$$

$$= (2xz - 2yz) + (x - 2xz) + (2yz - x)$$

$$= (2xz - 2xz) + (x - x) + (2yz - 2yz)$$

$$\text{div curl } \vec{F} = 0$$

Problem - 16

$\vec{F} = x^2y\,\vec{i} + y^2z\,\vec{j} + z^2x\,\vec{k}$. Find ~~curl curl~~

curl curl $\vec{F}$ (or) $\nabla \times (\nabla \times \vec{F})$.

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Solution :

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2y & y^2z & z^2x \end{vmatrix}$$

$$= \vec{i}\left[0 - y^2\right] - \vec{j}\left[z^2 - 0\right] + \vec{k}\left[0 - x^2\right]$$

$$\nabla \times \vec{F} = -y^2\,\vec{i} - z^2\vec{j} - x^2\vec{k}$$

$$\nabla \times (\nabla \times \vec{F}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y^2 & -z^2 & -x^2 \end{vmatrix}$$

$$= \vec{i}\left[0 + 2z\right] - \vec{j}\left[-2x - 0\right] + \vec{k}\left[0 + 2y\right]$$

$$= 2z\vec{i} + 2x\vec{j} + 2y\vec{k}$$

$$\nabla \times (\nabla \times \vec{F}) = 2(z\vec{i} + x\vec{j} + y\vec{k})$$

Problem - 17

Prove that curl$(\vec{r} \times \vec{a}) = -2\vec{a}$, where $\vec{a}$ is a constant vector.

Solution :

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$$

Let $\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$, where a_1, a_2, a_3 are constants.

$$\vec{r}\times\vec{a} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x & y & z \\ a_1 & a_2 & a_3 \end{vmatrix}$$

$$\vec{r}\times\vec{a} = \vec{i}(a_3y - a_2z) - \vec{j}(a_3x - a_1z) + \vec{k}(a_2x - a_1y)$$

$$\text{curl}(\vec{r}\times\vec{a}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_3y - a_2z & -a_3x + a_1z & a_2x - a_1y \end{vmatrix}$$

$$= \vec{i}\left[\frac{\partial}{\partial y}(a_2x - a_1y) - \frac{\partial}{\partial z}(-a_3x + a_1z)\right] - \vec{j}\left[\frac{\partial}{\partial x}(a_2x - a_1y) - \frac{\partial}{\partial z}(a_3y - a_2z)\right] + \vec{k}\left[\frac{\partial}{\partial x}(-a_3x + a_1z) - \frac{\partial}{\partial y}(a_3y - a_2z)\right]$$

$$= \vec{i}\left[-a_1 - a_1\right] - \vec{j}\left[a_2 + a_2\right] + \vec{k}\left[-a_3 - a_3\right]$$

$$= -2a_1\vec{i} - 2a_2\vec{j} - 2a_3\vec{k}$$

$$= -2\left[a_1\vec{i} + a_2\vec{j} + a_3\vec{k}\right]$$

$$= -2\vec{a}$$

$$\therefore \text{curl}(\vec{r}\times\vec{a}) = -2\vec{a}$$

Note:

i) $\nabla\cdot\vec{A} = \Sigma\left(\vec{i}\cdot\frac{\partial\vec{A}}{\partial x}\right)$

ii) $\nabla\times\vec{A} = \Sigma\left(\vec{i}\times\frac{\partial\vec{A}}{\partial x}\right)$

iii) $\vec{A}\cdot\nabla = \vec{A}\cdot\Sigma\vec{i}\frac{\partial}{\partial x}$

iv) $\vec{A}\times\nabla = \vec{A}\times\Sigma\vec{i}\frac{\partial}{\partial x}$

Book work:

1) $\nabla\cdot(\vec{A}+\vec{B}) = \nabla\cdot\vec{A} + \nabla\cdot\vec{B}$

2) $\nabla\cdot(\phi\vec{A}) = (\nabla\phi)\cdot\vec{A} + \cancel{\phi\times(\nabla\times\vec{A}})\ \phi(\nabla\cdot\vec{A})$

3) $\nabla\cdot(k\vec{A}) = k(\nabla\cdot\vec{A})$

4) $\nabla\times(\vec{A}+\vec{B}) = (\nabla\times\vec{A}) + (\nabla\times\vec{B})$

5) $\nabla\times(\phi\vec{A}) = (\nabla\phi)\times\vec{A} + \phi(\nabla\times\vec{A})$

6) $\nabla\times(k\vec{A}) = \vec{k}(\nabla\times\vec{A})$

i) Proof:

$$\nabla\cdot(\vec{A}+\vec{B}) = \Sigma\left(\vec{i}\cdot\frac{\partial}{\partial x}(\vec{A}+\vec{B})\right)$$

$$= \Sigma\vec{i}\cdot\left(\frac{\partial\vec{A}}{\partial x} + \frac{\partial\vec{B}}{\partial x}\right)$$

$$= \Sigma i\cdot\frac{\partial\vec{A}}{\partial x} + \Sigma\vec{i}\cdot\frac{\partial\vec{B}}{\partial x}$$

$$\nabla\cdot(\vec{A}+\vec{B}) = \nabla\cdot\vec{A} + \nabla\cdot\vec{B}$$

2) Proof:

$$\nabla \cdot (\phi \vec{A}) = \Sigma i \cdot \frac{\partial}{\partial x}(\phi \vec{A})$$

$$= \Sigma \vec{i} \cdot \left[\phi \frac{\partial \vec{A}}{\partial x} + \vec{A} \frac{\partial \phi}{\partial x}\right]$$

$$= \Sigma \left[\vec{i} \cdot \phi \frac{\partial \vec{A}}{\partial x} + \vec{i} \cdot \vec{A} \frac{\partial \phi}{\partial x}\right]$$

$$= \Sigma \vec{i} \cdot \phi \frac{\partial \vec{A}}{\partial x} + \Sigma \vec{i} \cdot \vec{A} \frac{\partial \phi}{\partial x}$$

$$= \cancel{\phi \Sigma \vec{i} \cdot \frac{\partial \vec{A}}{\partial x} + \vec{A} \Sigma \vec{i} \cdot \frac{\partial \phi}{\partial x}}$$

$$= \cancel{\phi(\nabla \cdot \vec{A})} +$$

$$= \phi\left(\Sigma \vec{i} \cdot \frac{\partial \vec{A}}{\partial x}\right) + \left(\Sigma \vec{i} \frac{\partial \phi}{\partial x}\right) \cdot \vec{A}$$

$$\nabla \cdot (\phi \vec{A}) = \phi(\nabla \cdot \vec{A}) + (\nabla \phi) \cdot \vec{A}$$

3) Proof:

$$\nabla \cdot (k\vec{A}) = \Sigma \vec{i} \cdot \frac{\partial}{\partial x}(k\vec{A})$$

$$= k\left(\Sigma \vec{i} \cdot \frac{\partial \vec{A}}{\partial x}\right)$$

$$\nabla \cdot (k\vec{A}) = k(\nabla \cdot \vec{A})$$

4) Proof:

$$\nabla \times (\vec{A} + \vec{B}) = \Sigma \vec{i} \times \frac{\partial}{\partial x}(\vec{A} + \vec{B})$$

$$= \Sigma \vec{i} \times \left(\frac{\partial \vec{A}}{\partial x} + \frac{\partial \vec{B}}{\partial x}\right)$$

$$= \left(\Sigma \vec{i} \times \frac{\partial \vec{A}}{\partial x}\right) + \left(\Sigma \vec{i} \times \frac{\partial \vec{B}}{\partial x}\right)$$

$$\nabla \times (\vec{A} + \vec{B}) = (\nabla \times \vec{A}) + (\nabla \times \vec{B})$$

5) Proof:

$$\nabla\times(\phi\vec{A}) = \Sigma\vec{i}\times\frac{\partial}{\partial x}(\phi\vec{A})$$

$$= \Sigma\vec{i}\times\left[\phi\frac{\partial\vec{A}}{\partial x} + \frac{\partial\phi}{\partial x}\vec{A}\right]$$

$$= \Sigma i\times\left(\phi\frac{\partial\vec{A}}{\partial x}\right) + \Sigma\vec{i}\times\left(\frac{\partial\phi}{\partial x}\vec{A}\right)$$

$$= \phi\left(\Sigma\vec{i}\times\frac{\partial\vec{A}}{\partial x}\right) + \cancel{\Sigma\vec{i}\times\frac{\partial\phi}{\partial x}}\left(\Sigma i\frac{\partial\phi}{\partial x}\right)\times\vec{A}$$

$$= \phi(\nabla\times\vec{A}) + (\nabla\phi)\times\vec{A}$$

$$\nabla\times(\phi\vec{A}) = (\nabla\phi)\times\vec{A} + \phi(\nabla\times\vec{A})$$

6) Proof:

$$\nabla\times(k\vec{A}) = \Sigma\vec{i}\times\frac{\partial}{\partial x}(k\vec{A})$$

$\frac{86}{\times 6}$ (

$$= \vec{k}\left(\Sigma\vec{i}\times\frac{\partial\vec{A}}{\partial x}\right)$$

$$\nabla\times(\vec{k}\vec{A}) = \vec{k}(\nabla\times\vec{A})$$

11/9/2020

Book work:

~~1. $\nabla(\vec{A}.\vec{B}) = \vec{A}\times(\nabla\times\vec{B}) + (\vec{A}.\nabla)\vec{B} + \vec{B}\times(\nabla<$~~

1) $\nabla(\vec{A}.\vec{B}) = \vec{A}\times(\nabla\times\vec{B}) + (\vec{A}.\nabla)\vec{B} + \vec{B}\times(\nabla\times\vec{A}) + (\vec{B}.\nabla)\vec{A}$

2) $\nabla\cdot(\vec{A}\times\vec{B}) = (\nabla\times\vec{A}).\vec{B} + (\nabla\times\vec{B}).\vec{A}$

3) $\nabla\times(\vec{A}\times\vec{B}) = \left\{(\vec{B}.\nabla)\vec{A} - (\nabla.\vec{A})\vec{B}\right\} - \left\{(\vec{A}.\nabla)\vec{B} - (\nabla.\vec{B})\vec{A}\right\}$

1) Proof:

$$\nabla(\vec{A}\cdot\vec{B}) = \sum\vec{i}\frac{\partial}{\partial x}(\vec{A}\cdot\vec{B})$$

$$= \sum\vec{i}\left[\frac{\partial\vec{A}}{\partial x}\cdot\vec{B} + \frac{\partial\vec{B}}{\partial x}\cdot\vec{A}\right]$$

$$= \sum\vec{i}\left(\frac{\partial A}{\partial x}\cdot\vec{B}\right) + \sum\vec{i}\left(\frac{\partial\vec{B}}{\partial x}\cdot\vec{A}\right) \longrightarrow ①$$

We know that,

$$\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c}$$

$$\therefore \vec{A}\times\left(\vec{i}\times\frac{\partial\vec{B}}{\partial x}\right) = \left(\vec{A}\cdot\frac{\partial\vec{B}}{\partial x}\right)\vec{i} - (\vec{A}\cdot\vec{i})\frac{\partial\vec{B}}{\partial x}$$

$$\Rightarrow \left(\vec{A}\cdot\frac{\partial\vec{B}}{\partial x}\right)\vec{i} = \vec{A}\times\left(\vec{i}\times\frac{\partial\vec{B}}{\partial x}\right) + (\vec{A}\cdot\vec{i})\frac{\partial\vec{B}}{\partial x}$$

① $\Rightarrow \sum\vec{A}\cdot$

$$\Rightarrow \sum\left(\vec{A}\cdot\frac{\partial\vec{B}}{\partial x}\right)\vec{i} = \sum\left[\vec{A}\times\left(\vec{i}\times\frac{\partial\vec{B}}{\partial x}\right) + (\vec{A}\cdot\vec{i})\frac{\partial\vec{B}}{\partial x}\right]$$

$$= \sum\vec{A}\times\left(\vec{i}\times\frac{\partial\vec{B}}{\partial x}\right) + \sum(\vec{A}\cdot\vec{i})\frac{\partial\vec{B}}{\partial x}$$

$$= \vec{A}\times\left(\sum\vec{i}\times\frac{\partial\vec{B}}{\partial x}\right) + \sum\left(\vec{A}\cdot\vec{i}\frac{\partial}{\partial x}\right)\vec{B}$$

$$\cancel{= \vec{A}\times\left(\sum\vec{i}\times\frac{\partial\vec{B}}{\partial x}\right) + \vec{A}\cdot\left(\sum\vec{i}\frac{\partial}{\partial x}\right)}$$

$$= \vec{A}\times\left(\sum\vec{i}\times\frac{\partial\vec{B}}{\partial x}\right) + \left(\vec{A}\cdot\sum\vec{i}\frac{\partial}{\partial x}\right)\vec{B}$$

$$\sum\left(\vec{A}\cdot\frac{\partial\vec{B}}{\partial x}\right)\vec{i} = \vec{A}\times(\nabla\cdot\vec{B}) + (\vec{A}\cdot\nabla)\vec{B}$$

$$\cancel{= \vec{A}\times(\nabla\cdot\vec{B}) + (\nabla\cdot\vec{A})\vec{B}}$$

Interchanging $\vec{A}$ and $\vec{B}$, we get

$$\sum\left(\vec{B}\cdot\frac{\partial\vec{A}}{\partial x}\right)\vec{i} = \vec{B}\times(\nabla\cdot\vec{A}) + (\vec{B}\cdot\nabla)\vec{A}$$

∴ From ①,

$$\nabla(\vec{A}\cdot\vec{B}) = \vec{A}\times(\nabla\cdot\vec{B}) + (\vec{A}\cdot\nabla)\vec{B} + \vec{B}\times(\nabla\cdot\vec{A}) + (\vec{B}\cdot\nabla)\vec{A}$$

2) Proof:

$$\nabla\cdot(\vec{A}\times\vec{B}) = \sum\vec{i}\cdot\frac{\partial}{\partial x}(\vec{A}\times\vec{B})$$

$$= \sum\vec{i}\cdot\left[\frac{\partial\vec{A}}{\partial x}\times\vec{B} + \vec{A}\times\frac{\partial\vec{B}}{\partial x}\right]$$

$$= \sum\vec{i}\cdot\left[\frac{\partial\vec{A}}{\partial x}\times\vec{B} - \frac{\partial\vec{B}}{\partial x}\times\vec{A}\right]$$

$$= \sum\vec{i}\cdot\left(\frac{\partial\vec{A}}{\partial x}\times\vec{B}\right) - \sum\vec{i}\cdot\left(\frac{\partial\vec{B}}{\partial x}\times\vec{A}\right)$$

$$= \sum\left(\vec{i}\times\frac{\partial\vec{A}}{\partial x}\right)\cdot\vec{B} - \sum\left(\vec{i}\times\frac{\partial\vec{B}}{\partial x}\right)\cdot\vec{A}$$

$$\nabla\cdot(\vec{A}\times\vec{B}) = \cancel{\nabla A}(\nabla\times\vec{A})\cdot\vec{B} - (\nabla\times\vec{B})\cdot\vec{A}$$

3. Proof:

$$\nabla\times(\vec{A}\times\vec{B}) = \sum\vec{i}\times\frac{\partial}{\partial x}(\vec{A}\times\vec{B})$$

$$= \sum\vec{i}\times\left[\frac{\partial\vec{A}}{\partial x}\times\vec{B} + \vec{A}\times\frac{\partial\vec{B}}{\partial x}\right]$$

$$= \sum\vec{i}\times\left(\frac{\partial\vec{A}}{\partial x}\times\vec{B}\right) - \sum\vec{i}\times\overset{\left(\frac{\partial\vec{B}}{\partial x}\times\vec{A}\right)}{\cancel{\left(\vec{A}\times\frac{\partial\vec{B}}{\partial x}\right)}} \longrightarrow ①$$

$$\sum\vec{i}\times\left(\frac{\partial\vec{A}}{\partial x}\times\vec{B}\right) = \sum\left\{(\vec{i}\cdot\vec{B})\frac{\partial\vec{A}}{\partial x} - \left(\vec{i}\cdot\frac{\partial\vec{A}}{\partial x}\right)\vec{B}\right\}$$

$$= \sum(\vec{i}\cdot\vec{B})\frac{\partial\vec{A}}{\partial x} - \sum\left(\vec{i}\cdot\frac{\partial\vec{A}}{\partial x}\right)\vec{B}$$

$$= \Sigma \left(\vec{B} \cdot \vec{i} \frac{\partial}{\partial x} \right) \vec{A} - \Sigma \left(\vec{i} \cdot \frac{\partial \vec{A}}{\partial x} \right) \vec{B}$$

$$= \left(\vec{B} \cdot \Sigma \vec{i} \frac{\partial}{\partial x} \right) \vec{A} - \left(\Sigma \left(\vec{i} \cdot \frac{\partial \vec{A}}{\partial x} \right) \right) \vec{B}$$

$$\Sigma \vec{i} \times \left(\frac{\partial \vec{A}}{\partial x} \times \vec{B} \right) = (\vec{B} \cdot \nabla) \vec{A} - (\nabla \cdot \vec{A}) \vec{B}$$

Interchanging $\vec{A}$ and $\vec{B}$, we get,

$$\Sigma \vec{i} \times \left(\frac{\partial \vec{B}}{\partial x} \times \vec{A} \right) = (\vec{A} \cdot \nabla) \vec{B} - (\nabla \cdot \vec{B}) \vec{A}$$

∴ ① becomes,

$$\nabla \times (\vec{A} \times \vec{B}) = \left\{ (\vec{B} \cdot \nabla) \vec{A} - (\nabla \cdot \vec{A}) \vec{B} \right\} - \left\{ (\vec{A} \cdot \nabla) \vec{B} - (\nabla \cdot \vec{B}) \vec{A} \right\}$$

Laplacian Operator:

The operator ∇^2 defined by

$$\nabla^2 = \nabla \cdot \nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$ is called Laplacian ~~operator~~. differential operator.

Definition:

If ϕ is such that $\nabla^2 \phi = 0$. Then ϕ is said to be harmonic function.

Note:

$$\nabla^2 = \Sigma \frac{\partial^2}{\partial x^2}$$

Book work

(x)

If a vector point function $\vec{A} = A_1\vec{i} + A_2\vec{j} + A_3\vec{k}$, where A_1, A_2, A_3 have continuous second order partials, then

i) $\nabla \cdot (\nabla \times \vec{A}) = 0$ (or) $\text{div}(\text{curl}\,\vec{A}) = 0$

ii) $\nabla \times (\nabla \times \vec{A}) = \nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A}$

Solution:

i) $\nabla \cdot (\nabla \times \vec{A}) = 0$

$$\nabla \times \vec{A} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_1 & A_2 & A_3 \end{vmatrix}$$

$$= \vec{i}\left[\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right] - \vec{j}\left[\frac{\partial A_3}{\partial x} - \frac{\partial A_1}{\partial z}\right] + \vec{k}\left[\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right]$$

$$\nabla \cdot (\nabla \times \vec{A}) = \nabla \cdot \Sigma \vec{i}\left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right)$$

$$= \Sigma \frac{\partial}{\partial x}\left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right)$$

$$= \Sigma\left(\frac{\partial^2 A_3}{\partial x \partial y} - \frac{\partial^2 A_2}{\partial x \partial z}\right)$$

$$= \frac{\partial^2 A_3}{\partial x \partial y} - \frac{\partial^2 A_2}{\partial x \partial z} - \frac{\partial^2 A_3}{\partial x \partial y} + \frac{\partial^2 A_1}{\partial y \partial z} + \frac{\partial^2 A_2}{\partial x \partial z} - \frac{\partial^2 A_1}{\partial y \partial z}$$

$$= 0$$

$$\therefore \nabla \cdot (\nabla \times \vec{A}) = 0$$

ii) $\nabla \times (\nabla \times \vec{A}) = \nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A}$.

$$\nabla \times \vec{A} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_1 & A_2 & A_3 \end{vmatrix}$$

$$= \vec{i}\left[\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right] - \vec{j}\left[\frac{\partial A_3}{\partial x} - \frac{\partial A_1}{\partial z}\right] + \vec{k}\left[\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right]$$

$$\nabla \times (\nabla \times \vec{A}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z} & \frac{\partial A_3}{\partial x} - \frac{\partial A_1}{\partial z} & \frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y} \end{vmatrix}$$

$$= \vec{i}\left[\frac{\partial}{\partial y}\left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right) - \frac{\partial}{\partial z}\left(\frac{\partial A_3}{\partial x} - \frac{\partial A_1}{\partial z}\right)\right] -$$

$$\vec{j}\left[\frac{\partial}{\partial x}\left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right) - \frac{\partial}{\partial z}\left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right)\right] +$$

$$\vec{k}\left[\frac{\partial}{\partial x}\left(\frac{\partial A_3}{\partial x} - \frac{\partial A_1}{\partial z}\right) - \frac{\partial}{\partial y}\left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right)\right]$$

$$= \sum \vec{i}\left[\frac{\partial}{\partial y}\left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right) - \frac{\partial}{\partial z}\left(\frac{\partial A_3}{\partial x} - \frac{\partial A_1}{\partial z}\right)\right]$$

$$\cancel{= \sum \vec{i} \left[\frac{\partial^2 A_2}{\partial y \partial x} - \frac{\partial^2 A_1}{\partial y^2} \right.}$$

$$= \sum \vec{i} \left[\frac{\partial^2 A_2}{\partial x \partial y} - \frac{\partial^2 A_1}{\partial y^2} + \frac{\partial^2 A_3}{\partial x \partial z} - \frac{\partial^2 A_1}{\partial z^2} \right]$$

$$= \sum \vec{i} \left[\frac{\partial^2 A_2}{\partial x \partial y} - \frac{\partial^2 A_1}{\partial y^2} + \frac{\partial^2 A_3}{\partial x \partial z} - \frac{\partial^2 A_1}{\partial z^2} - \frac{\partial^2 A_1}{\partial x^2} + \frac{\partial^2 A_1}{\partial x^2} \right]$$

$$= \sum \vec{i} \left[\frac{\partial^2 A_1}{\partial x^2} + \frac{\partial^2 A_2}{\partial x \partial y} + \frac{\partial^2 A_3}{\partial x \partial z} \right] - \sum \vec{i} \left[\frac{\partial^2 A_1}{\partial x^2} + \frac{\partial^2 A_1}{\partial y^2} + \frac{\partial^2 A_1}{\partial z^2} \right]$$

$$= \sum \vec{i} \cdot \frac{\partial}{\partial x} \left[\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z} \right] - \sum \vec{i}\, \nabla^2$$

$$= \sum \vec{i} \frac{\partial}{\partial x} (\nabla \cdot \vec{A}) - \nabla^2 \vec{A}$$

$$= \nabla (\nabla \cdot \vec{A}) - \nabla^2 \vec{A}$$

~~Book work:~~

$$\therefore \nabla \times (\nabla \times \vec{A}) = \nabla (\nabla \cdot \vec{A}) - \nabla^2 \vec{A}$$

17/9/2020

Book Work:

i) $\nabla \cdot (\nabla \phi) = \nabla^2 \phi$

ii) $\nabla \times (\nabla \phi) = 0.$

i) Proof:

$$\nabla \phi = \vec{i} \frac{\partial \phi}{\partial x} + \vec{j} \frac{\partial \phi}{\partial y} + \vec{k} \frac{\partial \phi}{\partial z}$$

$$\nabla\cdot(\nabla\phi) = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\cdot$$

$$\left(\vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}\right)$$

$$= \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}$$

$$= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\phi$$

$$= \nabla^2\phi$$

$$\therefore \nabla\cdot(\nabla\phi) = \nabla^2\phi$$

ii) Proof:

$$\nabla = \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}$$

$$\nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$$

$$\nabla\times(\nabla\phi) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial\phi}{\partial x} & \frac{\partial\phi}{\partial y} & \frac{\partial\phi}{\partial z} \end{vmatrix}$$

$$= \Sigma\vec{i}\left[\frac{\partial^2\phi}{\partial y\,\partial z} - \frac{\partial^2\phi}{\partial y\,\partial z}\right]$$

$$= \Sigma\vec{i}\,[0]$$

$$= 0$$

$$\therefore \nabla\times(\nabla\phi) = 0$$

Theorem

If $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ and $\vec{r} = r\hat{r}$.

Show that (i) $\nabla \cdot \left[f(r)\vec{r}\right] = rf'(r) + 3f(r)$

(ii) $\nabla \times \left[f(r)\vec{r}\right] = 0$

Proof:

i) We know that

$$\nabla \cdot (\phi\vec{A}) = (\nabla\phi)\cdot\vec{A} + \phi(\nabla\cdot\vec{A})$$

$$\therefore \nabla \cdot \left[f(r)\vec{r}\right] = \left[\nabla f(r)\right]\cdot\vec{r} + f(r)\left[\nabla\cdot\vec{r}\right] \longrightarrow ①$$

$$\nabla f(r) = \Sigma \vec{i} \frac{\partial}{\partial x}(f(r))$$

$$= \Sigma \vec{i} f'(r) \frac{\partial r}{\partial x}$$

$$= \Sigma \vec{i} f'(r) \frac{x}{r}$$

$$= \frac{f'(r)}{r} \Sigma \vec{i} x$$

$$= \frac{f'(r)}{r} (x\vec{i} + y\vec{j} + z\vec{k})$$

$$= \frac{f'(r)}{r} \vec{r}$$

$$= \frac{f'(r)}{r} r\hat{r}$$

$$\nabla f(r) = f'(r)\hat{r}$$

We have $\nabla\cdot\vec{r} = 3$

$$\nabla\cdot\vec{r} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z}$$

$$= 1 + 1 + 1$$

$$= 3$$

$\therefore$ ① becomes,

$$\nabla \cdot \left[f(r)\vec{r}\right] = \left[f'(r)\hat{r}\right] \cdot \vec{r} + f(r)(3)$$

$$~~= f'(r)\, r\vec{r}~~$$

$$= \left[f'(r)\hat{r}\right] \cdot r\hat{r} + 3f(r)$$

$$= r f'(r)\left[\hat{r}\cdot\hat{r}\right] + 3f(r)$$

$$= r f'(r)(1) + 3f(r)$$

$$= r f'(r) + 3f(r)$$

$$\therefore \nabla \cdot \left[f(r)\vec{r}\right] = r f'(r) + 3f(r)$$

ii) We know that

$$\nabla \times (\phi\vec{A}) = (\nabla\phi)\times\vec{A} + \phi(\nabla\times\vec{A})$$

$$\nabla \times \left[f(r)\vec{r}\right] = \left[\nabla f(r)\right]\times\vec{r} + f(r)\left[\nabla\times\vec{r}\right] \longrightarrow ①$$

We have

$$\nabla f(r) = f'(r)\hat{r} \text{ and } ~~\nabla \times f~~$$

$$\nabla \times \vec{r} = 0$$

$\therefore$ ① becomes,

$$\nabla\times\left[f(r)\vec{r}\right] = \left[f'(r)\hat{r}\right]\times\vec{r} + f(r)(0)$$

$$= f'(r)\hat{r} \times r\hat{r} + 0$$

$$= r f'(r)\left[\hat{r}\times\hat{r}\right]$$

$$= r f'(r)(\vec{0})$$

$$= \vec{0}$$

$$= 0$$

$$\therefore \nabla \times [f(r)\vec{r}] = 0$$

Corollary:

i) $\nabla \cdot (r^n \vec{r}) = (n+3) r^n$

ii) $\nabla \times (r^n \vec{r}) = 0$

Proof:

i) We have

$$\nabla \cdot [f(r)\vec{r}] = r f'(r) + 3 f(r) \longrightarrow ①$$

Take $f(r) = r^n$

Then $f'(r) = n r^{n-1}$

$$① \Rightarrow \nabla \cdot [r^n \vec{r}] = r[n r^{n-1}] + 3 r^n$$

$$= n r^n + 3 r^n$$

$$= (n+3) r^n$$

$$\therefore \nabla \cdot [r^n \vec{r}] = (n+3) r^n$$

ii) We have

$$\nabla \times [f(r)\vec{r}] = 0$$

Take $f(r) = r^n$

$$\therefore \nabla \times [r^n \vec{r}] = 0$$

Another Method.

i) $\nabla \cdot [r^n \vec{r}] = (n+3) r^n$

ii) $\nabla \times [r^n \vec{r}] = 0$

Proof:

i) $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

$r^n \vec{r} = r^n x \vec{i} + r^n y \vec{j} + r^n z \vec{k}$

$$\nabla \cdot [r^n \vec{r}] = \left[\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}\right] \cdot \left[r^n x \vec{i} + r^n y \vec{j} + r^n z \vec{k}\right]$$

$$= \frac{\partial}{\partial x}(r^n x) + \frac{\partial}{\partial y}(r^n y) + \frac{\partial}{\partial z}(r^n z)$$

$$= \Sigma \frac{\partial}{\partial x}(r^n x)$$

$$= \Sigma \left[r^n + x n r^{n-1} \frac{\partial r}{\partial x}\right]$$

$$= \Sigma \left[r^n + n x r^{n-1} \frac{x}{r}\right]$$

$$= \Sigma \left[r^n + n x^2 r^{n-2}\right]$$

$$= r^n + n r^{n-2} x^2 + r^n \, n r^{n-2} y^2 + r^n + n r^{n-2} z^2$$

$$= 3r^n + n r^{n-2}[x^2 + y^2 + z^2]$$

$$= 3r^n + n r^{n-2} r^2$$

$$= 3r^n + n r^n$$

$$= (3+n)\, r^n$$

$$\therefore \quad \nabla \cdot \left[r^n \vec{r} \right] = (n+3)\, r^n$$

ii)

$$\nabla = \vec{i} \frac{\partial}{\partial x} + \vec{j} \frac{\partial}{\partial y} + \vec{k} \frac{\partial}{\partial z}$$

$$r^n \vec{r} = r^n x\, \vec{i} + r^n y\, \vec{j} + r^n z\, \vec{k}$$

$$\nabla \times \left[r^n \vec{r} \right] = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ r^n x & r^n y & r^n z \end{vmatrix}$$

$$= \Sigma \vec{i} \left[\frac{\partial}{\partial y} (r^n z) - \frac{\partial}{\partial z} (r^n y) \right]$$

$$= \Sigma \vec{i} \left[z\, n r^{n-1} \frac{\partial r}{\partial y} - y\, n r^{n-1} \frac{\partial r}{\partial z} \right]$$

$$= \Sigma \vec{i} \left[z\, n r^{n-1} \frac{y}{r} - y n r^{n-1} \frac{z}{r} \right]$$

$$= \Sigma \vec{i} \left[n r^{n-2} zy - n r^{n-2} yz \right]$$

$$= 0$$

$$\therefore \quad \nabla \times \left[r^n \vec{r} \right] = 0$$

Problem - 18

If $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$. Find the value of n so that $r^n \vec{r}$ is solenoidal.

Solution:

We have $\nabla \cdot (r^n \vec{r}) = (n+3) r^n$

Given $r^n \vec{r}$ is solenoidal.

$\therefore \nabla \cdot (r^n \vec{r}) = 0$

$\Rightarrow (n+3) r^n = 0$

$\Rightarrow n+3 = 0$

$\Rightarrow n = -3$

Problem - 19

Prove that $\nabla \cdot \left(\frac{\vec{r}}{r^3}\right) = 0$

Proof:

$\vec{r} = x\vec{i} + y\vec{j} + \cancel{x} z\vec{k}$

$\therefore r^{-3}\vec{r} = r^{-3} x \vec{i} + r^{-3} y \vec{j} + r^{-3} \cancel{x} z \vec{k}$

$\nabla \cdot (r^{-3}\vec{r}) = \frac{\partial}{\partial x}(r^{-3}x) + \frac{\partial}{\partial y}(r^{-3}y) + \frac{\partial}{\partial z}(r^{-3}z)$

$= \Sigma \frac{\partial}{\partial x}(r^{-3}x)$

$= \Sigma \left[r^{-3} + x(-3) r^{-4} \frac{\partial r}{\partial x} \right]$

~~$= \Sigma\, r^{-3} - 3x r^{-4}$~~

$= \Sigma \left[r^{-3} - 3x\, r^{-4} \frac{x}{r} \right]$

$= \Sigma \left[r^{-3} - 3x^2 r^{-5} \right]$

$= r^{-3} - 3r^{-5}x^2 + r^{-3} - 3r^{-5}y^2 + r^{-3} - 3r^{-5}z^2$

$= 3r^{-3} - 3r^{-5}[x^2+y^2+z^2]$

$= 3r^{-3} - 3r^{-5}(r^2)$

$= 3r^{-3} - 3r^{-3}$

$\nabla \cdot (r^{-3}\vec{r}) = 0$

$\therefore \nabla \cdot \left(\frac{\vec{r}}{r^{-3}}\right) = 0$

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Problem - 20

If $\vec{v} = \vec{\omega} \times \vec{r}$, where $\vec{\omega}$ is a constant vector and $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ show that $\frac{1}{2}$ curl $\vec{v} = \vec{\omega}$

Solution:

$\vec{v} = \vec{\omega} \times \vec{r}$

Let $\vec{\omega} = \omega_1 \vec{i} + \omega_2 \vec{j} + \omega_3 \vec{k}$.

curl $\vec{v} = \nabla \times \vec{v}$

$= \Sigma \vec{i} \times \frac{\partial}{\partial x}(\vec{v})$

$= \Sigma \vec{i} \times \frac{\partial}{\partial x}(\vec{\omega} \times \vec{r})$

$= \Sigma \vec{i} \times (\vec{\omega} \times \vec{i})$

$= \Sigma \left[(\vec{i} \cdot \vec{i})\vec{\omega} - (\vec{i} \cdot \vec{\omega})\vec{i} \right]$

$= \Sigma \left[\vec{\omega} - (\vec{i} \cdot (\omega_1 \vec{i} + \omega_2 \vec{j} + \omega_3 \vec{k}))\vec{i} \right]$

$= \Sigma [\vec{\omega} - \omega_1 \vec{i}]$

$= \vec{\omega} - \omega_1 \vec{i} + \vec{\omega} - \omega_2 \vec{j} + \vec{\omega} - \omega_3 \vec{k}$

$= 3\vec{\omega} - [\omega_1 \vec{i} + \omega_2 \vec{j} + \omega_3 \vec{k}]$

$$= 3\vec{\omega} - \vec{\omega}$$

$$\text{curl } \vec{v} = 2\vec{\omega}$$

$$\frac{1}{2} \text{curl } \vec{v} = \vec{\omega}$$

Problem - 21

Show that $\nabla^2 r^n = n(n+1) r^{n-2}$, where n is constant.

Solution:

$$\nabla^2 r^n = \frac{\partial^2 r^n}{\partial x^2} + \frac{\partial^2 r^n}{\partial y^2} + \frac{\partial^2 r^n}{\partial z^2}$$

$$\frac{\partial^2 r^n}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial r^n}{\partial x}\right)$$

$$= \frac{\partial}{\partial x}\left[n r^{n-1} \frac{\partial r}{\partial x}\right]$$

$$= \frac{\partial}{\partial x}\left[n r^{n-1} \frac{x}{r}\right]$$

$$= \frac{\partial}{\partial x} \cancel{nr^{n}x} \left[n r^{n-2} x\right]$$

$$= n \frac{\partial}{\partial x}\left[r^{n-2} x\right]$$

$$= n\left[r^{n-2} + x(n-2) r^{n-3} \frac{\partial r}{\partial x}\right]$$

$$= n\left[r^{n-2} + x(n-2) r^{n-3} \frac{x}{r}\right]$$

$$= n\left[r^{n-2} + (n-2) r^{n-4} x^2\right].$$

$$\frac{\partial^2 r^n}{\partial x^2} = n r^{n-2} + n(n-2) r^{n-4} x^2$$

$$\nabla^2 r^n = nr^{n-2} + n(n-2)r^{n-4}x^2 + nr^{n-2} + n(n-2)r^{n-4}y^2$$

$$+ nr^{n-2} + n(n-2)r^{n-4}z^2$$

$$= 3nr^{n-2} + n(n-2)r^{n-4}[x^2+y^2+z^2]$$

$$= 3nr^{n-2} + n(n-2)r^{n-4}r^2$$

$$= 3nr^{n-2} + n(n-2)r^{n-2}$$

~~$= nr^{n-2}[3 + n(n-2$~~

$$= nr^{n-2}[3+n-2]$$

$$= nr^{n-2}(n+1)$$

$$\nabla^2 r^n = n(n+1)r^{n-2}$$

Part - A

1) $\nabla \cdot \vec{r} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 1+1+1 = 3$

1. If $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$, then div $\vec{r}$ is

(a) 0 (b) 1 (c) 2 [✓ (d) 3]

2) $\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} =$

$= \frac{\partial}{\partial x}(2xy^3z^4) + \frac{\partial}{\partial y}(3x^2y^2z^4) + \frac{\partial}{\partial z}(4x^2y^3z^3)$

$= 2y^3z^4 + 6x^2yz^4 + 12x^2y^3z^2$

2) If $\phi = x^2y^3z^4$, then $\nabla^2\phi$ is

(a) $2xy^2z^3 + 6x^2yz^3 - 12x^2y^3z^2$

(b) $2y^3z^4 + 6x^2yz^4 + x^2y^3z^2$

[✓ (c) $2y^3z^4 + 6x^2yz^4 + 12x^2y^3z^2$]

(d) $2y^3z^4 + x^2yz^4 + 12x^2y^3z^2$

3) If $\nabla \times \vec{v} = 0$ then v is

(a) solenoidal [✓ (b) irrotational]

(c) 0 (d) none

$$\nabla \times \vec{r} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix}$$

$= \vec{i}\left(\frac{\partial z}{\partial y} - \frac{\partial y}{\partial z}\right) \vec{j}\left(\frac{\partial z}{\partial x} - \frac{\partial x}{\partial z}\right)$

$+ \vec{k}\left(\frac{\partial y}{\partial x} - \frac{\partial x}{\partial y}\right) = 0$

$\nabla \times A = 0$

$(\nabla \cdot (\nabla \times \vec{A})) = 0$

4) $\nabla \cdot (\nabla \times \vec{A})$ is

(a) 1 [✓ (b) 0] (c) 3 (d) none

5) Curl of gradient of a vectors is $\nabla \times (\nabla \phi)$

(a) unity [✓ (b) zero] (c) Null vector

(d) depends on the constants of the vector.

6) Divergence of the vector $y\vec{i} + z\vec{j} + x\vec{k}$ $\frac{\partial y}{\partial x} + \frac{\partial z}{\partial y} + \frac{\partial x}{\partial z}$ $= 0$

(a) -1 [✓ (b) 0] (c) 1 (d) 3

7) If $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$, then $\nabla \cdot \vec{r}$ is $\frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z}$

$1+1+1$

$=3$

(a) 0 (b) 1 (c) 2 [✓ (d) 3]

8) The unit vector corresponding to $2\vec{i} + 2\vec{j} - \vec{k}$ is

(a) $\frac{2\vec{i} + 2\vec{j} - \vec{k}}{5}$ [✓ (b) $\frac{2\vec{i} + 2\vec{j} - \vec{k}}{3}$]

(c) $\frac{2\vec{i} + 2\vec{j} - \vec{k}}{2}$ (d) $\frac{2\vec{i} + 2\vec{j} - \vec{k}}{4}$

9) The vector $(4xy - z^3)\vec{i} - 3xz^2\vec{k}$ is

(a) solenoidal (b) irrotational

(c) both solinoidal irrotational [✓ (d) neither solenoidal nor irrotational]

10) If $\vec{A} = z\vec{i} + x\vec{j} + y\vec{k}$ then $(\nabla \times \vec{A})$ is

(a) $x\vec{i} + y\vec{j} + z\vec{k}$ [✓ (b) $\vec{i} + \vec{j} + \vec{k}$]

(c) $y\vec{i} + z\vec{j} + x\vec{k}$ (d) none of these.

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z & x & y \end{vmatrix} = \vec{i}\left(\frac{\partial y}{\partial y} - \frac{\partial x}{\partial z}\right) - \vec{j}\left(\frac{\partial y}{\partial x} - \frac{\partial z}{\partial z}\right) + \vec{k}\left(\frac{\partial x}{\partial x} - \frac{\partial z}{\partial y}\right)$$

$$= \vec{i}(1) - \vec{j}(-1) + \vec{k}(1)$$

$$= \vec{i} + \vec{j} + \vec{k}$$

11) The second degree ~~or~~ laplacial differential equation is

(a) $\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)^2$ [✓ (b)] $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$

(c) $\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y}$ (c) $\frac{\partial}{\partial x} + \frac{\partial}{\partial y}$

Unit - 3

Line integrals and surface integrals

Line integral:

Let (x, y, z) be any point on whose parametric equations are

$$x = x(t) \quad , \quad y = y(t) \quad , \quad z = z(t)$$

and the vector equation is

$$\vec{r} = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}$$

Let the end points of ~~of~~ c be A, B given by $t = t_1$, $t = t_2$

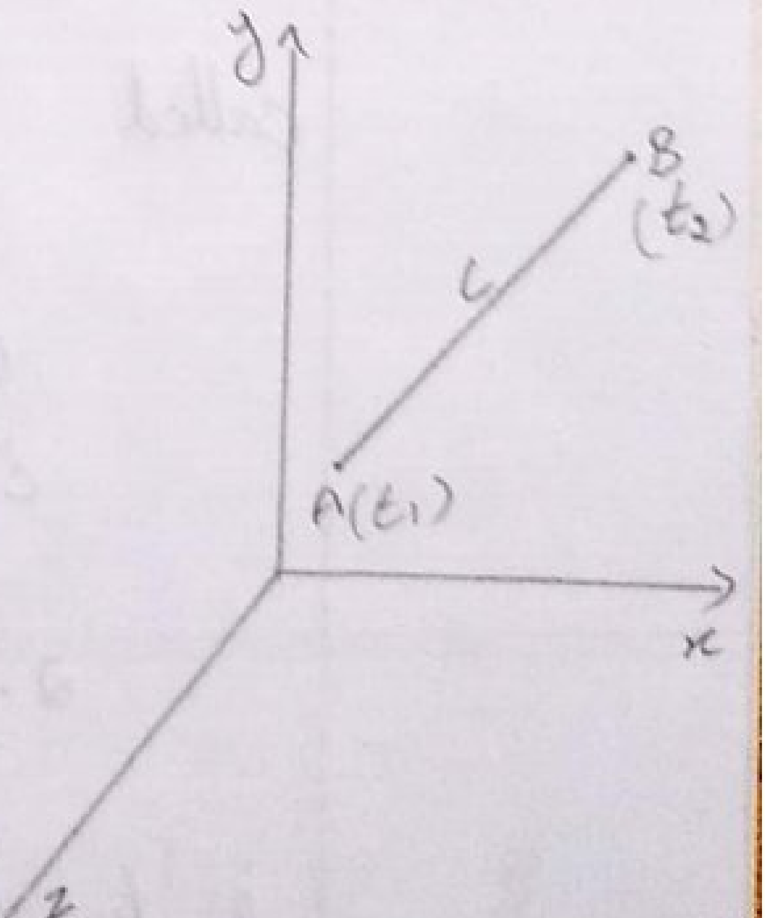
Let a vector point function

$\vec{f}(x, y, z) = f_1\vec{i} + f_2\vec{j} + f_3\vec{k}$ be defined on c.

Then the integral

$$\int_{t_1}^{t_2} \vec{f}\cdot d\vec{r} = \int_c \vec{f}\cdot d\vec{r}$$

$$= \int_c (f_1\vec{i} + f_2\vec{j} + f_3\vec{k}).$$
$$(dx\,\vec{i} + dy\,\vec{j} + dz\,\vec{k})$$



$$= \int_C (f_1 dx + f_2 dy + f_3 dz)$$

is called line integral $\vec{f}$ over c.

Note:

The other forms of line integral are

$$\int_C \phi \, dr = \int_C \vec{f} \times d\vec{r}$$

Working rule to evaluate $\int_C \vec{f} \cdot d\vec{r}$:

Obtain the parametric equation in t or c and the parametric values t_1 and t_2 for the end points of c.

Get $\vec{f} \cdot d\vec{r}$ in the form $\phi(t)dt$ where $d\vec{r} = dx\vec{i} + dy\vec{j} + dz\vec{k}$

Evaluate $\int_{t_1}^{t_2} \phi(t) \, dt$.

Note:

1. If c is closed curve, then $\int_C \vec{f} \cdot d\vec{r}$ is called ~~an~~ circulation and is denoted by

$$\oint_C \vec{f} \cdot d\vec{r}$$

2. Work done by the force $\vec{f}$ along c from A to B $= \int_A^B \vec{f} \cdot d\vec{r}$.

Problem - 1

Find the value of integral $\int_C \vec{A}\cdot\vec{dr}$, where $\vec{A} = yz\vec{i} + zx\vec{j} - xy\vec{k}$ along c whose parametric equations are $x = t$, $y = t^2$, $z = t^3$ drawn from $O(0,0,0)$ to $P(2,4,8)$

Solution:

Parametric equations are

$$x = t \quad , \quad y = t^2 \quad , \quad z = t^3$$

$$dx = dt \quad , \quad dy = 2t\,dt \quad , \quad dz = 3t^2 dt$$

$$\vec{A}\cdot\vec{dr} = (yz\vec{i} + zx\vec{j} - xy\vec{k})\cdot(dx\vec{i} + dy\vec{j} + dz\vec{k})$$

$$= yz\,dx + zx\,dy - xy\,dz$$

$$= t^2 t^3 dt + t^3 t\,(2t\,dt) - t t^2 (3t^2 dt)$$

$$= t^5 dt + 2t^5 dt - 3t^5 dt$$

$$= 3t^5 dt - 3t^5 dt$$

$$= (3t^5 - 3t^5)\,dt$$

$$\vec{A}\cdot\vec{dr} = 0$$

$$\int_C \vec{A}\cdot\vec{dr} = \int_0^2 0$$

$$= 0$$

24/9/2020 Problem - 2

Find the value of the integral $\int_C \vec{F}\cdot\vec{dr}$ if $\vec{F} = 3xy\vec{i} - y^2\vec{j}$ and c is the curve

$x = t \ , \ y = 2t^2$ from $(0,0)$ to $(1,2)$

~~Soln~~

Solution :

$$\vec{F} = 3xy\,\vec{i} - y^2\,\vec{j}$$

$$x = t \ , \ y = 2t^2$$

$$dx = dt \ , \ dy = 4t\,dt$$

$$\vec{F}\cdot\vec{dr} = (3xy\,\vec{i} - y^2\vec{j})\cdot(dx\,\vec{i} + dy\,\vec{j})$$

$$= 3xy\,dx - y^2 dy$$

~~$= 3t(2t^2)\,dt - 2t^2(4t\,dt)$~~

~~$= 6t^3 dt - 8t^3\,dt$~~

$$= 3(t)(2t^2)\,dt - 4t^4(4t\,dt)$$

$$= 6t^3\,dt - 16t^5\,dt$$

$$= (6t^3 - 16t^5)\,dt$$

$x = 0 \Rightarrow t = 0$

$x = 1 \Rightarrow t = 1$

$$\therefore \int_C \vec{F}\cdot\vec{dr} = \int_0^1 (6t^3 - 16t^5)\,dt$$

$$= \left[6\frac{t^4}{4} - 16\frac{t^6}{6}\right]_0^1$$

$$= \left[\frac{3t^4}{2} - \frac{8t^6}{3}\right]_0^1$$

$$= \left[\frac{3}{2} - \frac{8}{3}\right] - [0 - 0]$$

$$= \frac{9 - 16}{6}$$

$$= -7/6$$

$$\int_C \vec{F}\cdot d\vec{r} = -7/6$$

Problem - 3

Evaluate integral $\int_C \vec{F}\cdot d\vec{r}$. If $\vec{F} = (3x^2+6y)\vec{i} - 14yz.\vec{j} + 20xz^2\vec{k}$ and c is the curve $x=t$, $y=t^2$, $z=t^3$ from $(0,0,0)$ to $(1,1,1)$.

~~Soln~~

Solution:

$$\vec{F} = (3x^2+6y)\vec{i} - 14yz\vec{j} + 20xz^2\vec{k}$$

$$x=t \quad , \quad y=t^2 \quad , \quad z=t^3$$

$$dx = dt \quad , \quad dy = 2t\,dt \quad , \quad dz = 3t^2 dt$$

$$\vec{F}\cdot d\vec{r} = \left((3x^2+6y)\vec{i} - 14yz\vec{j} + 20xz^2\vec{k}\right)\cdot (dx\,\vec{i} + dy\,\vec{j} + dz\,\vec{k})$$

$$= (3x^2+6y)dx - 14yz\,dy + 20xz^2 dz$$

$$= (3t^2+6t^2)dt - 14t^2t^3(2t\,dt) + \cancel{20tt^6}\; 20t\,t^6(3t^2dt)$$

$$= 9t^2 dt - 28t^6 dt + 60t^9 dt$$

$$= (9t^2 - 28t^6 dt + 60t^9)dt$$

$$x=0 \Rightarrow t=0$$

$$x=1 \Rightarrow t=1$$

$$\int_0^1 \vec{F}\cdot d\vec{r} = \int_0^1 (9t^2 - 28t^6 dt + 60t^9)\,dt$$

$$= \left[\frac{9t^3}{3} - \frac{28t^7}{7} + \frac{60t^{10}}{10}\right]_0^1$$

$$= \left[3t^3 - 4t^7 + 6t^{10}\right]_0^1$$

$$= (3 - 4 + 6) - (0)$$

$$= 5$$

$$\int_0^1 \vec{F} \cdot \vec{dr} = 5$$

25/9/2020

Problem - 4

Evaluate integral $\int_C \vec{F} \cdot \vec{dr}$ along $y^2 = 4x$ from $(0,0)$ to $(4,4)$ where $\vec{F} = x\vec{j} - y\vec{i}$

~~Soln~~

Solution:

$$\vec{F} = -y\vec{i} + x\vec{j}$$

Given curve is $y^2 = 4x$

(i.e) $x = t^2$, $y = 2t$

$$dx = 2t\,dt \quad , \quad dy = 2dt$$

$$\vec{F} \cdot \vec{dr} = (-y\vec{i} + x\vec{j}) \cdot (dx\,\vec{i} + dy\,\vec{j})$$

$$= -y\,dx + x\,dy$$

$$= -2t(2t\,dt) + t^2(2\,dt)$$

$$= -4t^2\,dt + 2t^2\,dt$$

$$\vec{F} \cdot \vec{dr} = -2t^2\,dt$$

$x = 0 \Rightarrow t = 0$

$x = 4 \Rightarrow t = 2$

$$\therefore \int_C \vec{F} \cdot \vec{dr} = \int_0^2 -2t^2\,dt$$

$$= -2 \int_0^2 t^2\,dt$$

$$= -2\left[\frac{t^3}{3}\right]_0^2$$

$$= -2\left[\frac{8}{3} - 0\right]$$

$$= \frac{-16}{3}$$

$$\int_0^2 \vec{F}\cdot\vec{dr} = \frac{-16}{3}$$

Another method for problem-4.

$$\vec{F} = -y\vec{i} + x\vec{j}$$

Given curve is $y^2 = 4x$

$$2y\,dy = 4\,dx$$

$$y\,dy = 2\,dx$$

$$\vec{F}\cdot\vec{dr} = (-y\vec{i} + x\vec{j}).(dx\,\vec{i} + dy\,\vec{j})$$

$$= -y\,dx + x\,dy$$

~~$= -y\left(\frac{y\,dy}{2}\right) + x\left(\frac{2dx}{y}\right)$~~

$$= -y\left(\frac{y\,dy}{2}\right) + \frac{y^2}{4}\,dy$$

$$= \frac{-1}{2}\,y^2\,dy + \frac{1}{4}\,y^2\,dy$$

~~$= \left[\frac{-1}{2}y^2 + \frac{1}{4}y^2\right]dy$~~

$$= \left[\frac{-1}{2} + \frac{1}{4}\right] y^2\,dy$$

$$= \left[\frac{-4+2}{8}\right] y^2\,dy$$

$$= \left(-\tfrac{2}{8}\right) y^2\,dy$$

$\vec{F} \cdot \vec{dr} = \frac{-1}{4} y^2 dy$

$\int_c \vec{F} \cdot \vec{dr} = \frac{-1}{4} \int_0^4 y^2 \, dy$

$= \frac{-1}{4} \left[\frac{y^3}{3} \right]_0^4$

$= \frac{-1}{12} \left[(4)^3 - 0 \right]$

$= \frac{-1}{\cancel{12}_3} (\overset{16}{\cancel{64}})$

$\int_c \vec{F} \cdot \vec{dr} = \frac{-16}{3}$

Problem - 5

Evaluate the integral $\int_c \vec{F} \cdot \vec{dr}$. If $\vec{F} = (5xy - 6x^2)\vec{i} + (2y - 4x)\vec{j}$ and c is the arc of the parabola $y = x^3$ for $(1,1)$ to $(2,8)$ then show that $\int_c \vec{F} \cdot \vec{dr} = 35$

Solution:

Given $y = x^3$

$dy = 3x^2 \, dx$

$\vec{F} \cdot \vec{dr} = \left((5xy - 6x^2)\vec{i} + (2y - 4x)\vec{j} \right) \cdot \left(dx\,\vec{i} + dy\,\vec{j} \right)$

$= (5xy - 6x^2) dx + (2y - 4x) dy$

$= (5x x^3 - 6x^2) dx + (2x^3 - 4x)(3x^2 dx)$

$= (5x^4 - 6x^2) dx + (6x^5 - 12x^3) dx$

$\vec{F} \cdot \vec{dr} = \left[5x^4 - 6x^2 + 6x^5 - 12x^3 \right] dx$

$$\int_C \vec{F}\cdot\vec{dr} = \int_1^2 \left[5x^4 - 6x^2 + 6x^5 - 12x^3\right] dx$$

$$= \left[5\frac{x^5}{5} - 6\frac{x^3}{3} + 6\frac{x^6}{6} - 12\frac{x^4}{4}\right]_1^2$$

$$= \left[x^5 - 2x^3 + x^6 - 3x^4\right]_1^2$$

$$= [32 - 16 + 64 - 48] - [1 - 2 + 1 - 3]$$

$$= [96 - 64] - [-3]$$

$$= 32 + 3$$

$$= 35$$

$$\therefore \int_C \vec{F}\cdot\vec{dr} = 35$$

Problem - 6

Evaluate the integral $I = \int_C (x\,dx + y\,dy + z\,dz)$ where c is the circle $x^2 + y^2 + z^2 = a^2$ and $z = 0$.

Solution :

Equation of the circle is

$$x^2 + y^2 + z^2 = a^2 \; ; \; z = 0$$

(i.e) $x = a\cos\theta$ and $y = a\sin\theta$, $z = 0$

Then $dx = -a\sin\theta\, d\theta$, $dy = a\cos\theta\, d\theta$, $dz = 0$

$$\vec{F}\cdot\vec{dr} = x\,dx + y\,dy + z\,dz$$

$$= (a\cos\theta)(-a\sin\theta\, d\theta) + (a\sin\theta)(a\cos\theta\, d\theta) + 0$$

$$= -a^2\sin\theta\cos\theta\, d\theta + a^2\sin\theta\cos\theta\, d\theta$$

$$\vec{F}\cdot\vec{dr} = 0$$

$$\therefore \int_C \vec{F} \cdot \vec{dr} = 0.$$

26/09/2020

Problem - 7

Evaluate $\int_C x\,dx + y\,dy$, where C is the ellipse $x^2 + 4y^2 = 4$.

Solution :

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Given equation is

$$x^2 + 4y^2 = 4.$$

$x = a\cos\theta$,

$y = a\sin\theta$

$$\frac{x^2}{4} + \frac{y^2}{1} = 1$$

$$\therefore \quad x = 2\cos\theta \qquad y = \sin\theta$$

$$dx = -2\sin\theta\, d\theta, \quad dy = \cos\theta\, d\theta$$

θ various from 0 to 2π

$$\vec{F} \cdot \vec{dr} = x\,dx + y\,dy$$

$$= (2\cos\theta)(-2\sin\theta\, d\theta) + \sin\theta\,(\cos\theta\, d\theta)$$

$$= -4\sin\theta\cos\theta\, d\theta + \sin\theta\cos\theta\, d\theta$$

$$\vec{F} \cdot \vec{dr} = -3\sin\theta\cos\theta\, d\theta$$

$$\int_C \vec{F} \cdot \vec{dr} = -3\int_0^{2\pi} \sin\theta\cos\theta\, d\theta$$

$\sin 2\theta = 2\sin\theta\cos\theta$

$\sin\theta\cos\theta = \frac{\sin 2\theta}{2}$

$$= -3\int_0^{2\pi} \frac{\sin 2\theta}{2}\, d\theta$$

$$= \frac{-3}{2}\left[\frac{-\cos 2\theta}{2}\right]_0^{2\pi}$$

$$= \frac{3}{4}(\cos 4\pi - \cos 0)$$

$$= \frac{3}{4}(1-1)$$

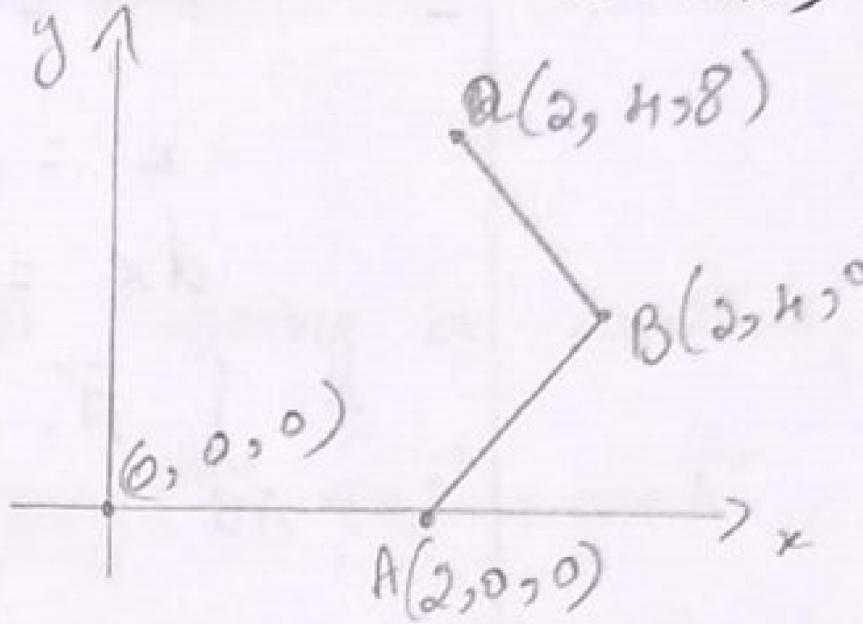
$$= 0$$

$$\int_C \vec{F} \cdot \vec{dr} = 0$$

Problem - 8

Find the value of the integral $\int_C \vec{A} \cdot \vec{dr}$, where $\vec{A} = yz\vec{i} + zx\vec{j} - xy\vec{k}$ and c is the curve obtained by joining 0 to A(2, 0, 0) then A(2, 0, 0) to B(2, 4, 0) and then B to Q(2, 4, 8) by straight lines.

Solution:



$$\vec{A} \cdot \vec{dr} = (yz\vec{i} + zx\vec{j} - xy\vec{k}) \cdot (dx\vec{i} + dy\vec{j} + dz\vec{k})$$

$$= yz\,dx + zx\,dy - xy\,dz$$

Equation of the straight line is

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1} = t$$

$$\int_C \vec{A} \cdot \vec{dr} = \int_{OA} \vec{A} \cdot \vec{dr} + \int_{AB} \vec{A} \cdot \vec{dr} + \int_{BQ} \vec{A} \cdot \vec{dr} \longrightarrow ①$$

Along OA $\quad O(0, 0, 0) \quad A(2, 0, 0)$

$(x_1, y_1, z_1) \quad (x_2, y_2, z_3)$

$$\frac{x - 0}{2 - 0} = \frac{y - 0}{0 - 0} = \frac{z - 0}{0 - 0} = t$$

$$\frac{x}{2} = t, \quad y = 0, \quad z = 0$$

$$x = 2t, \quad y = 0, \quad z = 0$$

$$dx = 2dt, \quad dy = 0, \quad dz = 0$$

$$\therefore \int_{OA} \vec{A} \cdot d\vec{r} = \int_{OA} 0 = 0$$

Along AB $A(2,0,0)$ $B(2,4,0)$
(x_1, y_1, z_1) (x_2, y_2, z_2)

$$\frac{x-2}{2-2} = \frac{y-0}{4-0} = \frac{z-0}{0-0} = t$$

$$x-2 = 0, \quad \frac{y}{4} = t, \quad z = 0$$

~~$x = t+2$~~

$$x = 2, \quad y = 4t, \quad z = 0$$
$$dx = 0, \quad dy = 4dt, \quad dz = 0$$

$$\therefore \int_{AB} \vec{A} \cdot d\vec{r} = \int_{AB} 0$$

$$= 0$$

Along BQ $B(2,4,0)$ $Q(2,4,8)$
(x_1, y_1, z_1) (x_2, y_2, z_2)

$$\frac{x-2}{2-2} = \frac{y-4}{4-4} = \frac{z-0}{8-0} = t$$

$$x-2 = 0, \quad y-4 = 0, \quad z = 8t$$

$$x = 2, \quad y = 4, \quad z = 8t$$

$$dx = 0, \quad dy = 0, \quad dz = 8dt$$

$$\int_{BQ} \vec{A} \cdot d\vec{r} = \int_{BQ} 0 + 0 - xy\, dz$$

$$= \int_0^1 -(2)(4)8\, dt$$

~~$= \int$~~

$$= -64 \int_0^1 dt$$

$$= -64 [t]_0^1$$

$= -64$

$$\therefore \int_{BA} \vec{A}\cdot d\vec{r} = -64$$

$\therefore$ ① becomes.

$$\int_C \vec{A}\cdot d\vec{r} = 0 + 0 - 64$$

$$\therefore \int_C \vec{A}\cdot d\vec{r} = -64$$

Problem - 9

Find the work done is moving a particle in a force field $\vec{F} = 3xy\vec{i} - 5z\vec{j} + 10x\vec{k}$ along the curve $x = t^2+1$, $y = 2t^2$, $z = t^3$ from $(2,2,1)$ to $(5,8,8)$

Solution:

$$\vec{F}\cdot d\vec{r} = (3xy\vec{i} - 5z\vec{j} + 10x\cdot\vec{k})\cdot(dx\,\vec{i} + dy\,\vec{j} + dz\,\vec{k})$$

$$= 3xy\,dx - 5z\,dy + 10x\,dz$$

$x = t^2+1$, $y = 2t^2$, $z = t^3$

$dx = 2t\,dt$, $dy = 4t\,dt$, $dz = 3t^2\,dt$

$$\therefore \vec{F}\cdot d\vec{r} = 3(t^2+1)(2t^2)(2t\,dt) - 5t^3(4t\,dt) + 10(t^2+1)(3t^2dt)$$

$$= 6t(2t^4+2t^2)\,dt - 20t^4\,dt + 30t^2(t^2+1)\,dt$$

$$= (12t^5+12t^3)dt - 20t^4\,dt + (30t^4+30t^2)dt$$

$$\vec{F}\cdot d\vec{r} = (12t^5 + 10t^4 + 12t^3 + 30t^2)\,dt$$

$$\int_C \vec{F}\cdot\vec{dr} = \int_1^2 \left[12t^5 + 10t^4 + 12t^3 + 30t^2\right] dt$$

$$= \left[12\frac{t^6}{6} + 10\frac{t^5}{5} + 12\frac{t^4}{4} + 30\frac{t^3}{3}\right]_1^2$$

$$= \left[2t^6 + 2t^5 + 3t^4 + 10t^3\right]_1^2$$

$$= \left[2(64) + 2(32) + 3(16) + 10(8)\right] - \left[2+2+3+10\right]$$

$$= 128 + 64 + 48 + 80 - 17$$

$$= 320 - 17$$

$$= 303$$

$$\therefore \int_C \vec{F}\cdot\vec{dr} = 303$$

Problem - 10

Find the work done in moving a particle in a forced field $\vec{F} = 3x^2\vec{i} + (2xz - y)\vec{j} + z\vec{k}$ along the straight line $(0,0,0)$ $(2,1,3)$.

Solution:

$$\vec{F} = 3x^2\vec{i} + (2xz - y)\vec{j} + z\vec{k}$$

$$\vec{dr} = dx\,\vec{i} + dy\,\vec{j} + dz\,\vec{k}$$

$$\vec{F}\cdot\vec{dr} = 3x^2\,dx + (2xz - y)\,dy + z\,dz$$

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1} = t$$

$(0,0,0)$ and $(2,1,3)$

$$\frac{x-0}{2-0} = \frac{y-0}{1-0} = \frac{z-0}{3-0} = t$$

$$\frac{x}{2} = t \;,\; \frac{y}{1} = t \;,\; \frac{z}{3} = t$$

$$x = 2t \;,\; y = t \;,\; z = 3t$$

$$dx = 2dt \;,\; y = dt \;,\; z = 3dt$$

$$\vec{F}\cdot\vec{dr} = 3(4t^2)(2dt) + (2(2t)(3t) - t)\,dt + 3t\,(3dt)$$

~~$= 3(12)t^2\,dt + (12t^2 - t)\,dt + 9t\,dt$~~

$$= 24t^2\,dt + (12t^2 - t)\,dt + 9t\,dt$$

$$= (24t^2 + 12t^2 - t + 9t)\,dt$$

$$\vec{F}\cdot\vec{dr} = (36t^2 + 8t)\,dt$$

$$\int \vec{F}\cdot\vec{dr} = \int_0^1 (36t^2 + 8t)\,dt$$

$$= \left[36\frac{t^3}{3} + 8\frac{t^2}{2}\right]_0^1$$

$$= \left[12t^3 + 4t^2\right]_0^1$$

$$= (12 + 4) - (0)$$

$$= 16$$

$$\therefore \int \vec{F}\cdot\vec{dr} = 16$$

(X) Book work:

03/10/2020

The necessary and sufficient condition for the integral $\int_{C(A_1,A_2)} \vec{f}\cdot\vec{dr}$ to be independent of path of integration is the existence of a scalar point function ϕ such that $\vec{f} = \nabla\phi$.

Proof:

Necessary Part

Given that the line integral depends on the end points alone.

We have to prove that there exists a scalar point function ϕ such that $\vec{f} = \nabla\phi$

Suppose that $\vec{f}$ is defined in D and the symbol (A_1, P) ~~deper~~ denotes any curve in D joining A_1 and P.

If ~~$P(x_1, y_1$.~~ $P(x, y, z)$ is a variable point in D, then the integral $\int_{(A_1,P)} \vec{f} \cdot d\vec{r} \rightarrow ①$ depends on P and not on the curve to (A_1, P).

Hence the integral ① defines a scalar point function in D.

Let the function be denoted by $\phi(P)$, that is $\phi(x, y, z)$

Then,

~~$\phi(x, y, z) = \int_{(A_1, P)} \vec{f} \cdot d\vec{r}$~~

~~$= \int$~~

$$\phi(x, y, z) = \int_{(A_1, P)} \vec{f} \cdot d\vec{r}$$

$$= \int_{(x_1, y_1, z_1)}^{(x, y, z)} \vec{f} \cdot d\vec{r}$$

$$= \int_{(x_1, y_1, z_1)}^{(x, y, z)} \vec{f} \cdot \vec{T} \, ds$$

where $\vec{T}$ is a unit vector along an arbitrary chosen curve through (x_1, y_1, z_1) and (x, y, z)

Now, $\frac{d\phi}{ds} = \vec{f} \cdot \vec{T}$ (or)

$$(\nabla\phi) \cdot \vec{T} = \vec{f} \cdot \vec{T}$$

(i.e) $(\nabla\phi - \vec{f}) \cdot \vec{T} = 0$

Since $\vec{T}$ is arbitrary, $\nabla\phi - \vec{f} = 0$

$$\therefore \vec{f} = \nabla\phi$$

~~Sf~~

Sufficient Part:

Given there exists a scalar functions ϕ such that $\vec{f} = \nabla\phi$

To prove that the line integral is independent of path.

Let c be arbitrary curve with the end points (x_1, y_1, z_1) and (x_2, y_2, z_2)

$$\int_c \vec{f} \cdot \vec{dr} = \int_c (\nabla\phi) \cdot \vec{dr}$$

$$= \int_c \left(\vec{i} \frac{\partial\phi}{\partial x} + \vec{j} \frac{\partial\phi}{\partial y} + \vec{k} \frac{\partial\phi}{\partial z} \right) \cdot (dx \vec{i} + dy \vec{j} + dz \vec{k})$$

$$= \int_c \left(\frac{\partial\phi}{\partial x} dx + \frac{\partial\phi}{\partial y} dy + \frac{\partial\phi}{\partial z} dz \right)$$

$$= \int_C d\phi$$

$$= \left[\phi\right]_{(x_1, y_1, z_1)}^{(x_2, y_2, z_2)}$$

$$= \phi(x_2, y_2, z_2) - \phi(x_1, y_1, z_1)$$

which is independent of c.

Definition:

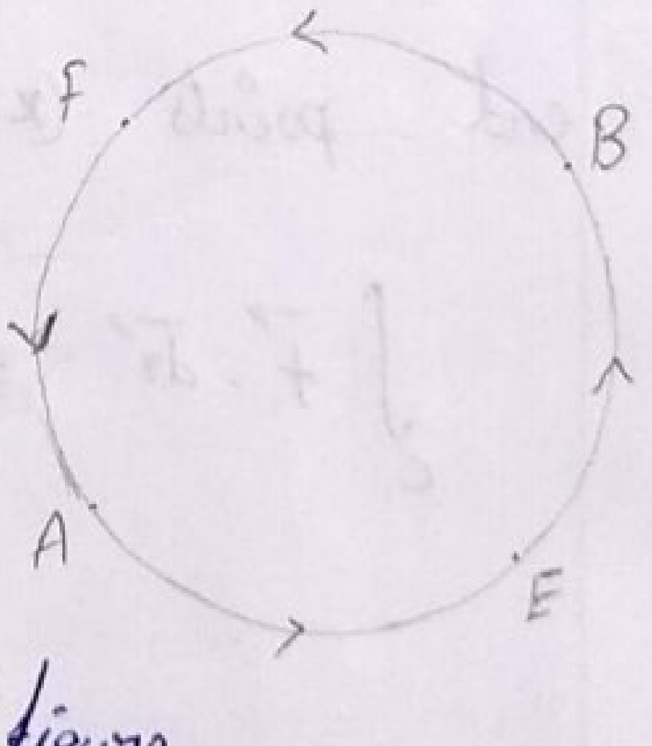
If a vector field $\vec{f}$ is such that there exists a scalar point function, ϕ such that $\vec{f} = \nabla\phi$. Then $\vec{f}$ is said to be conservative field and ϕ is said to be its scalar potential.

Theorem

In a conservative field $\vec{f}$. $\int_C \vec{f} \cdot d\vec{r} = 0$ where c is any simple closed curve.

Proof:

Let A, B, E, F be points on c taken in which c is oriented as shown as in the following figure.



$$\int_C = \int_{AEB} + \int_{BFA} = \int_{AEB} - \int_{AFB} = \int_{AEB} - \int_{AEB} = 0$$

6/10/2020

Problem - 11

Show that the line integral of $\vec{F} = (3x^2+6xy)\vec{i} + (3x^2-y^2)\vec{j}$ is independent of path of integration. Find $\int \vec{F}\cdot d\vec{r}$ along any curve joining (0,0) and (1,2)

Solution :

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 3x^2+6xy & 3x^2-y^2 & 0 \end{vmatrix}$$

$$= \vec{i}\left[\frac{\partial}{\partial y}(0) - \frac{\partial}{\partial z}(3x^2-y^2)\right] - \vec{j}\left[\frac{\partial}{\partial x}(0) - \frac{\partial}{\partial z}(3x^2+6xy)\right] + \vec{k}\left[\frac{\partial}{\partial x}(3x^2-y^2) - \frac{\partial}{\partial y}(3x^2+6xy)\right]$$

$$= \vec{i}(0-0) - \vec{j}(0-0) + \vec{k}(6x - 6x)$$

$$= \vec{0}$$

$\therefore \vec{F} \times d\vec{r}$ is independent of path of integration.

Straight line joining (0,0) and (1,2) is

$$\frac{x-0}{1-0} = \frac{y-0}{2-0} = t \qquad \frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = t$$

$$\frac{x}{1} = \frac{y}{2} = t$$

$\therefore x = t \quad , \quad y = 2t$

$dx = dt \quad , \quad dy = 2dt$

$$\vec{F}\cdot d\vec{r} = (3x^2+6xy)\,dx + (3x^2-y^2)\,dy$$

$$= (3t^2+6(t)(2t))\,dt + (3t^2-(2t)^2)(2dt)$$

$$= (3t^2+12t^2)\,dt + (6t^2-8t^2)dt$$

$$= (15t^2-2t^2)\,dt$$

$$\vec{F}\cdot d\vec{r} = 13t^2\,dt$$

$$\int \vec{F}\cdot d\vec{r} = \int_0^1 13t^2\,dt \qquad \begin{aligned} y=0 &\Rightarrow t=0 \\ y=2 &\Rightarrow t=1 \end{aligned}$$

$$= 13\left[\frac{t3}{3}\right]_0^1$$

$$\int \vec{F}\cdot d\vec{r} = \frac{13}{3}$$

Problem - 12

Prove that the vector field

$\vec{F} = (y+y^2+z^2)\vec{i} + (x+z+2xy)\vec{j} + (y+2zx)\vec{k}$ is conservative and find its scalar potential.

Solution:

$$\nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \frac{\partial}{\partial y} & \partial/\partial z \\ y+y^2+z^2 & x+z+2xy & y+2zx \end{vmatrix}$$

$$= \vec{i}\left[\frac{\partial}{\partial y}(y+2zx) - \frac{\partial}{\partial z}(x+z+2xy)\right] -$$

$$\vec{j}\left[\frac{\partial}{\partial x}(y+2zx) - \frac{\partial}{\partial z}(y+y^2+z^2)\right] +$$

$$\vec{k}\left[\frac{\partial}{\partial x}(x+z+2xy) - \frac{\partial}{\partial y}(y+y^2+z^2)\right]$$

$$= \vec{i}\left[(1+0)-(0+1+0)\right] - \vec{j}\left[2z-2z\right] + \vec{k}\left[2y+1-1-2y\right]$$

$$= \vec{i}(0) - \vec{j}(0) + \vec{k}(0)$$

$$= \vec{0}$$

$\therefore \vec{f}$ is conservative vector field.

Let ϕ be the scalar potential.

Then $\nabla\phi = \vec{f}$

$$\vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z} = (y+y^2+z^2)\vec{i} + (x+z+2xy)\vec{j} + (y+2zx)\vec{k}$$

$$\therefore \frac{\partial\phi}{\partial x} = y+y^2+z^2 \longrightarrow ①$$

$$\frac{\partial\phi}{\partial y} = x+\overset{z}{\cancel{y}}+2xy \longrightarrow ②$$

$$\frac{\partial\phi}{\partial z} = y+2zx \longrightarrow ③$$

Integrating ① with respect to x, we get,

$$\int\partial\phi = \int(y+y^2+z^2)\,\cancel{dz}\,\partial x$$

$$\phi = xy + xy^2 + xz^2 + f(y,z) \longrightarrow ④$$

Integrating ② with respect to y, we get,

$$\int\partial\phi = \int(x+\overset{z}{\cancel{y}}+2xy)\,\partial y$$

~~$\phi = xy + \frac{y^2}{2}$~~

$$\phi = xy + yz + 2x\frac{y^2}{2} + \cancel{\phi}\,g(x,z)$$

$$\phi = xy + yz + xy^2 + g(x,z) \longrightarrow ⑤$$

Integrating ③ with respect to z, we get,

$$\int \partial\phi = \int (y+2zx)\,dz$$

$$\phi = yz + 2x\frac{z^2}{2} + h(x,y)$$

$$\phi = yz + xz^2 + h(x,y) \longrightarrow ⑥$$

∴ From ④, ⑤ and ⑥, we get

$$\phi(x,y,z) = xy + xy^2 + xz^2 + yz + c.$$

6/10/2020

Surface Integral

1) $$\iint_S \phi\, ds = \iint_{R_{xy}} \phi \frac{dx\,dy}{|\hat{n}\cdot\vec{k}|}$$

2) $$\iint_S \phi\, ds = \iint_{R_{yz}} \phi \frac{dy\,dz}{|\hat{n}\cdot\vec{i}|}$$

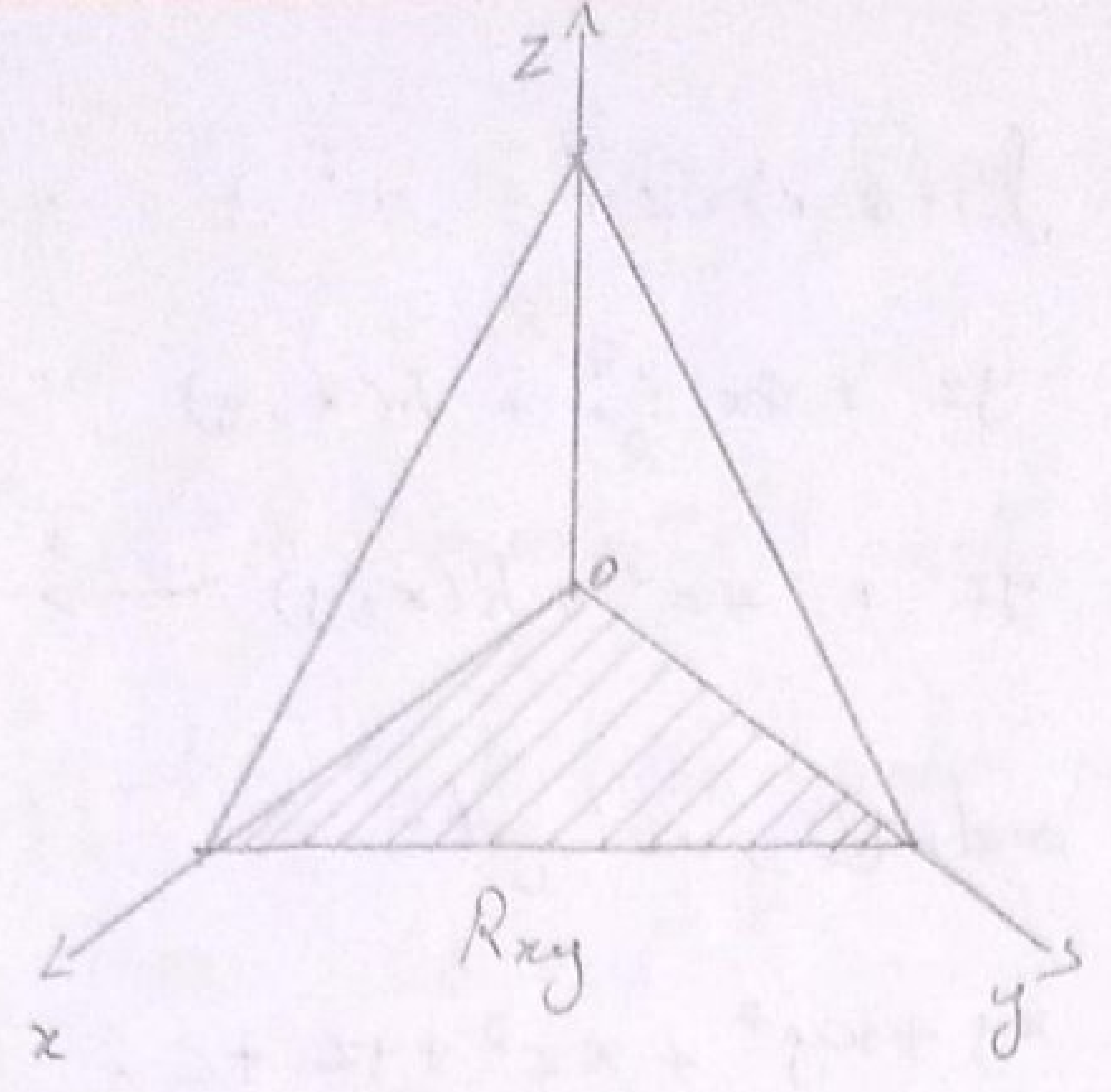
3) $$\iint_S \phi\, ds = \iint_{R_{zx}} \phi \frac{dz\,dx}{|\hat{n}\cdot\vec{j}|}$$

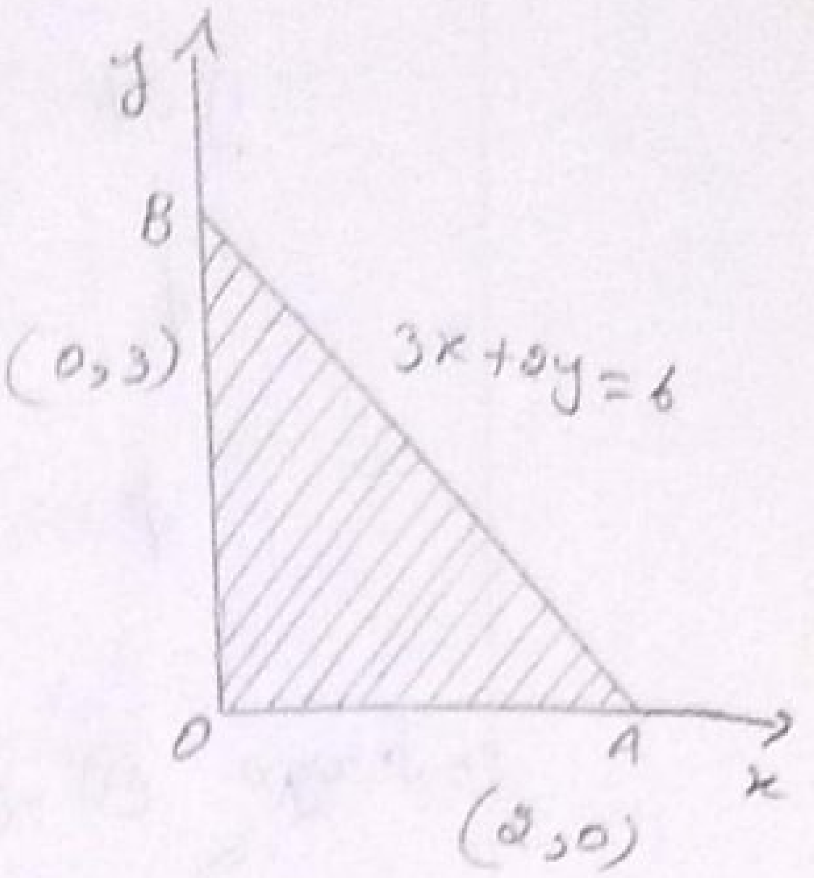
where R_{xy}, R_{yz}, R_{zx} are projections of S on xoy plane, yoz plane, zox plane respectively.

Problem - 13

Evaluate the integral $\iint_S \vec{A}\cdot\hat{n}\, ds$ if $\vec{A} = 4y\vec{i} + 18z\vec{j} - x\vec{k}$ and S is the surface of the portion of the plane $3x+2y+6z=6$ contained in the first octant.

Solution:





$$\phi = 3x + 2y + 6z - 6$$

$$\nabla\phi = 3\vec{i} + 2\vec{j} + 6\vec{k}$$

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{3\vec{i} + 2\vec{j} + 6\vec{k}}{\sqrt{9+4+36}} = \frac{3\vec{i} + 2\vec{j} + 6\vec{k}}{\sqrt{49}}$$

$$= \frac{3\vec{i} + 2\vec{j} + 6\vec{k}}{7} = \frac{3}{7}\vec{i} + \frac{2}{7}\vec{j} + \frac{6}{7}\vec{k}$$

Let R_{xy} be the projection s on xoy plane.

$$\iint_S \vec{A}\cdot\hat{n}\, ds = \iint_{R_{xy}} \vec{A}\cdot\hat{n}\, \frac{dx\, dy}{|\hat{n}\cdot\vec{k}|}$$

$$\vec{A}\cdot\hat{n} = (4y\vec{i} + 18z\vec{j} - x\vec{k})\cdot\left(\frac{3}{7}\vec{i} + \frac{2}{7}\vec{j} + \frac{6}{7}\vec{k}\right)$$

$$= (4y)(3/7) + (18z)\left(\frac{2}{7}\right) - (x)(6/7)$$

$$= \cancel{4}\ \frac{1}{7}\left[12y + 36z - 6x\right]$$

$$= \frac{1}{7}\left[12y + 36\left(\frac{6-3x-2y}{6}\right) - 6x\right]$$

$$= \frac{1}{7}\left[12y + 6(6-3x-2y) - 6x\right]$$

$$= \frac{1}{7}\left[12y + 36 - 18x - 12y - 6x\right]$$

$$= \frac{1}{7}\left[36 - 24x\right]$$

$$= \frac{12}{7}\left[3 - 2x\right]$$

$$\hat{n} \cdot \vec{k} = \frac{6}{7}$$

$$|\hat{n} \cdot \vec{k}| = 6/7$$

$$\iint_S \vec{A} \cdot \hat{n}\, ds = \int_0^2 \int_0^{\frac{6-3x}{2}} \frac{12/7\,(3-2x)}{6/7}\, dx\, dy$$

$$= \int_0^2 \int_0^{\frac{6-3x}{2}} 2(3-2x)\, dx\, dy$$

$$= 2\int_0^2 (3-2x)\, dx \left[y\right]_0^{\frac{6-3x}{2}}$$

$$= 2\int_0^2 (3-2x)\, \cancel{d} \left(\frac{6-3x}{2}\right) dx$$

$$= \frac{2}{2}\int_0^2 (18 - 9x - 12x + 6x^2)\, dx$$

$$= \int_0^2 (18 - 21x + 6x^2)\, dx$$

$$= \left[18x - 21\frac{x^2}{2} + \frac{6x^3}{3}\right]_0^2$$

$$= \left[18x - \frac{21}{2}x^2 + 2x^3\right]_0^2$$

$$= 18(2) - \frac{21}{2}(4) + 2(8)$$

$$= 36 - 42 + 16$$

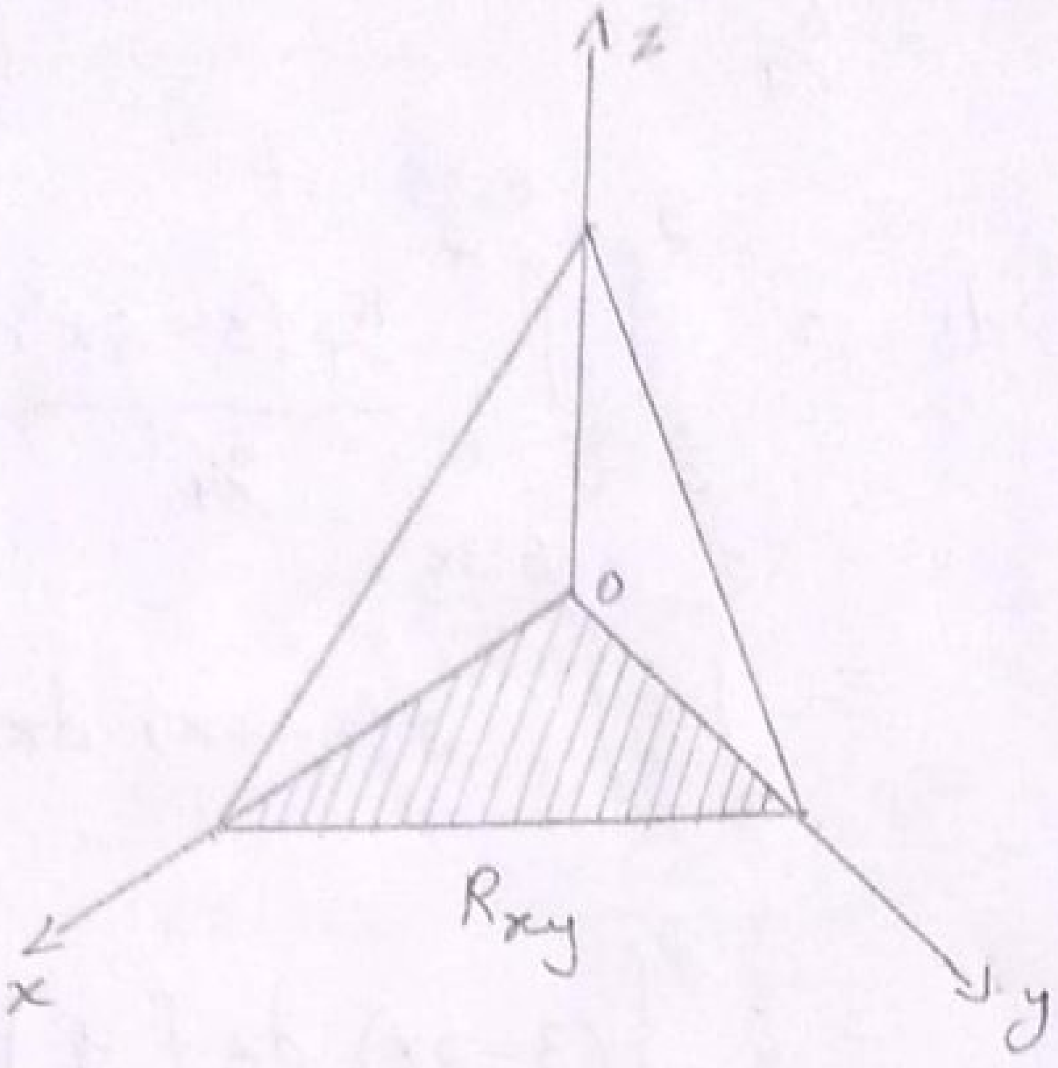
$$= 10$$

10/10/2020

Problem - 14

Evaluate $\iint_S \vec{A}\cdot\hat{n}\,ds$ if $\vec{A} = (x+y^2)\vec{i} - 2x\vec{j} + 2yz\vec{k}$ and s is the surface of plane $2x + y + 2z = 6$ in the first octant.

Solution:



$$\phi = 2x + y + 2z - 6$$

$$\nabla\phi = 2\vec{i} + \vec{j} + 2\vec{k}$$

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2\vec{i}+\vec{j}+2\vec{k}}{\sqrt{4+1+4}} = \frac{2\vec{i}+\vec{j}+2\vec{k}}{\sqrt{9}}$$

$$= \frac{2\vec{i}+\vec{j}+2\vec{k}}{3} = \frac{2}{3}\vec{i} + \frac{1}{3}\vec{j} + \frac{2}{3}\vec{k}$$

Let R_{xy} be the projection to S on ~~the~~ xoy plane.

$$\iint_S \vec{A}\cdot\hat{n}\,ds = \iint_{R_{xy}} \vec{A}\cdot\hat{n}\,\frac{dx\,dy}{|\hat{n}\cdot\vec{k}|}$$

$$\vec{A}\cdot\hat{n} = (x+y^2)(2/3) - (2x)(1/3) + (2yz)(2/3)$$

$$= \frac{2}{3}\left[x + y^2 - x + 2yz\right]$$

$$= 2/3\left[y^2 + 2yz\right]$$

$$= \frac{2}{3}\left[y^2 + 2y\left(\frac{6-2x-y}{2}\right)\right]$$

$$= \frac{2}{3}\left[y^2 + 6y - 2xy - y^2\right]$$

$$= \frac{2}{3}\left[6y - 2xy\right]$$

$$\hat{n}\cdot\vec{k} = \frac{2}{3}$$

$$|\hat{n}\cdot\vec{k}| = \frac{2}{3}$$

$$\frac{\vec{A}\cdot\hat{n}}{|\hat{n}\cdot\vec{k}|} = \frac{2}{3}(6y - 2xy) \times \frac{3}{2}$$

$$= 6y - 2xy$$

$$\iint_S \vec{A}\cdot\hat{n}\, ds = \int_0^3 \int_0^{6-2x} (6y - 2xy)\, dx\, dy$$

$$= 2\int_0^3 \int_0^{6-2x} (3y - xy)\, dx dy$$

$$= 2\int_0^3 \int_0^{6-2x} (3-x)(y)\, dx\, dy$$

$$= 2\int_0^3 (3-x)\, dx \left[\frac{y^2}{2}\right]_0^{6-2x}$$

$(3-x)(9+x^2-6x)$

$$= \int_0^3 (3-x)\,(6-2x)^2\, dx$$

$18 + 3x^2 - 18x$

$-9x - x^3 + 6x^2$

$$= \int_0^3 (3-x)\; 4(3-x)^2\, dx$$

$18 - 27x + 9x^2 - x^3$

$18 - 27(3) + 9(9)$

$-27 - \cancel{18}$

$$= 4\int_0^3 (3-x)^3\, dx$$

$-\cancel{81} + \cancel{81} - 27$

= [illegible]

$$= 4\left[\frac{(3-x)^4}{(-1)(4)}\right]_0^3$$

$$= -1\left[(3-3)^4 - (3-0)^4\right]$$

$$= -\left[0 - 81\right]$$

$$= 81$$

~~Problem~~-15.

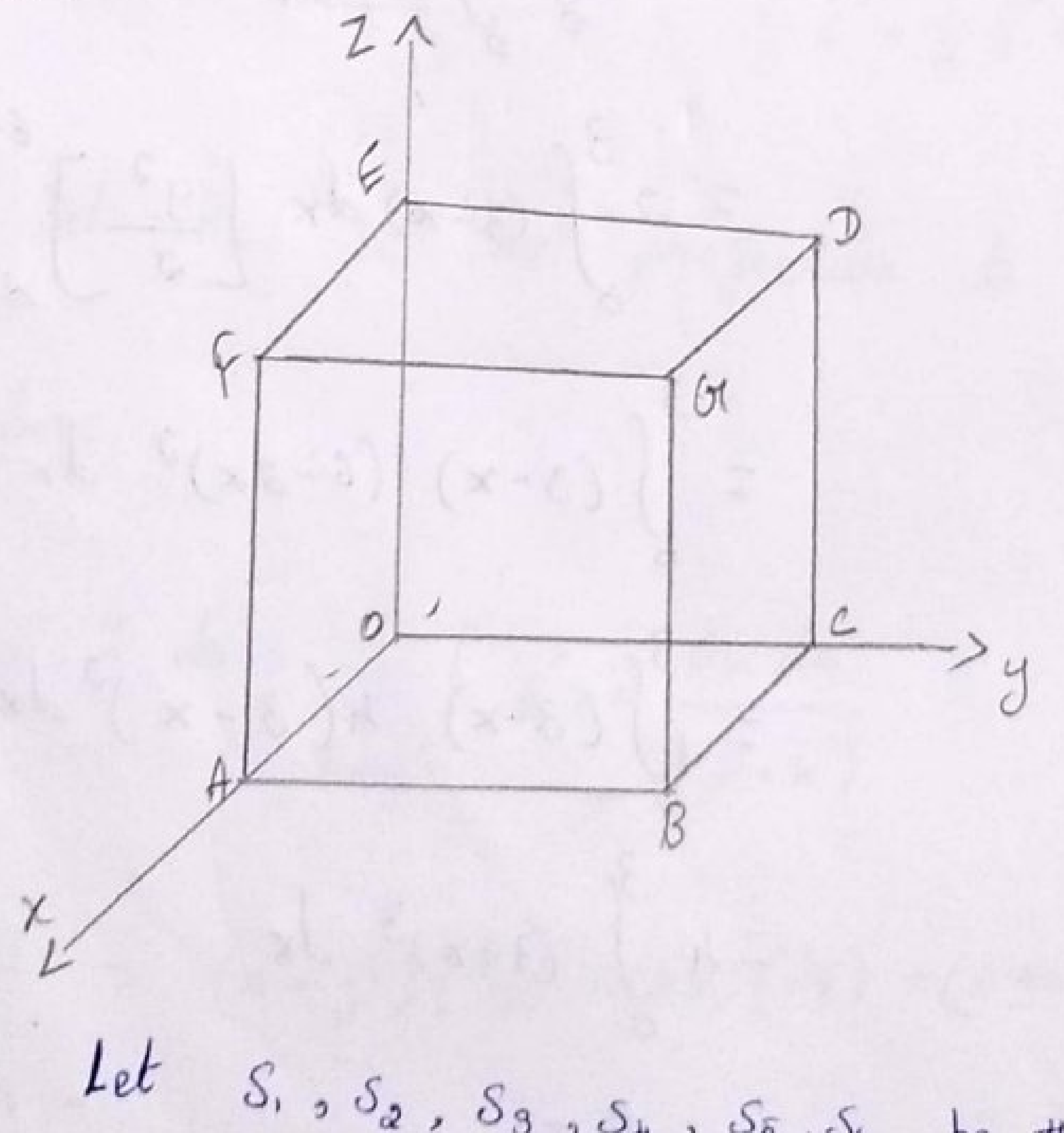
~~Evaluate~~ $\cancel{\iint_S \vec{F}\cdot\vec{dr}}$ $\cancel{\iint_S \vec{F}\cdot\hat{n}\, ds}$ ~~if~~

$\cancel{\vec{F}} = \cancel{(x+y)}\, i$

Problem -15

$\iint_S \vec{F}\cdot\hat{n}\, ds$ if $\vec{F} = (x+y)\vec{i} + x\vec{j} + z\vec{k}$ and S is the surface of the cube bounded by the planes $x=0$, $x=1$, $y=0$, $y=1$, $z=0$, $z=1$.

Solution:



Let $S_1, S_2, S_3, S_4, S_5, S_6$ be the surfaces corresponding to the planes $x=0$, $x=1$, $y=0$, $y=1$, $z=0$ and $z=1$.

$$\therefore \iint_S \vec{F}\cdot\hat{n}\,ds = \iint_{S_1} + \iint_{S_2} + \iint_{S_3} + \iint_{S_4} + \iint_{S_5} + \iint_{S_6}$$

$$[\vec{F}\cdot\hat{n}]\,ds. \longrightarrow ①$$

On S_1

$x=0 , \quad \hat{n} = -\vec{i}$

$$\frac{\vec{F}\cdot\hat{n}}{|\hat{n}\cdot\vec{i}|} = \frac{-(x+y)}{1} = -y$$

$$\iint_{S_1} \vec{F}\cdot\hat{n}\,ds = \int_0^1\int_0^1 -y\,dy\,dz$$

$$= -\left[\tfrac{y^2}{2}\right]_0^1 [z]_0^1$$

$$= -\left[\tfrac{1}{2}\right][1]$$

$$= -\tfrac{1}{2}$$

On S_2

$x=1 , \quad \hat{n} = \vec{i}$

$$\frac{\vec{F}\cdot\hat{n}}{|\hat{n}\cdot\vec{i}|} = \frac{(x+y)}{1} = 1+y$$

$$\iint_{S_2} \vec{F}\cdot\hat{n}\,ds = \int_0^1\int_0^1 (1+y)\,dy\,dz$$

$$= \left[y+\frac{y^2}{2}\right]_0^1 [z]_0^1$$

$$= \left[1+\tfrac{1}{2}\right][1]$$

$$= \tfrac{3}{2}$$

On S_3 $\quad y=0$, $\hat{n} = -\vec{j}$

$$\frac{\vec{F}\cdot\hat{n}}{|\hat{n}\cdot\vec{j}|} = \frac{-x}{1} = -x$$

$$\iint_{S_3} \vec{F}\cdot\hat{n}\, ds = \int_0^1\int_0^1 (-x)\, dx\, dz$$

$$= -\left[\frac{x^2}{2}\right]_0^1 [z]_0^1$$

$$= -\left(\frac{1}{2}\right)(1)$$

$$= \frac{-1}{2}$$

On S_4

$y=1$, $\hat{n} = \vec{j}$

$$\frac{\vec{F}\cdot\hat{n}}{|\hat{n}\cdot\vec{j}|} = x$$

$$\iint_{S_4} \vec{F}\cdot\hat{n}\, ds = \int_0^1\int_0^1 x\, dx\, dz$$

$$= \left[\frac{x^2}{2}\right]_0^1 [z]_0^1$$

$$= \left(\frac{1}{2}\right)(1)$$

$$= \frac{1}{2}$$

On S_5

$z=0$, $\hat{n} = -\vec{k}$

$$\frac{\vec{F}\cdot\hat{n}}{|\hat{n}\cdot\vec{k}|} = \frac{-z}{1} = 0$$

Scanned with CamScanner

$$\iint_{S_5} \vec{F} \cdot \hat{n}\, ds = 0$$

On S_6

$z = 1$, $\hat{n} = \vec{k}$

$$\frac{\vec{F} \cdot \hat{n}}{|\hat{n} \cdot \vec{k}|} = \frac{z}{1} = 1$$

$$\iint_{S_6} \vec{F} \cdot \hat{n}\, ds = \int_0^1 \int_0^1 dx\, dy$$

$$= [x]_0^1 \, [y]_0^1$$

$$= (1)(1)$$

$$= 1$$

$\therefore$ ① becomes

$$\iint_S \vec{F} \cdot \hat{n}\, ds = \frac{-1}{2} + \frac{3}{2} - \frac{1}{2} + \frac{1}{2} + 0 + 1$$

$$= \frac{2}{2} + 1 = 1 + 1$$

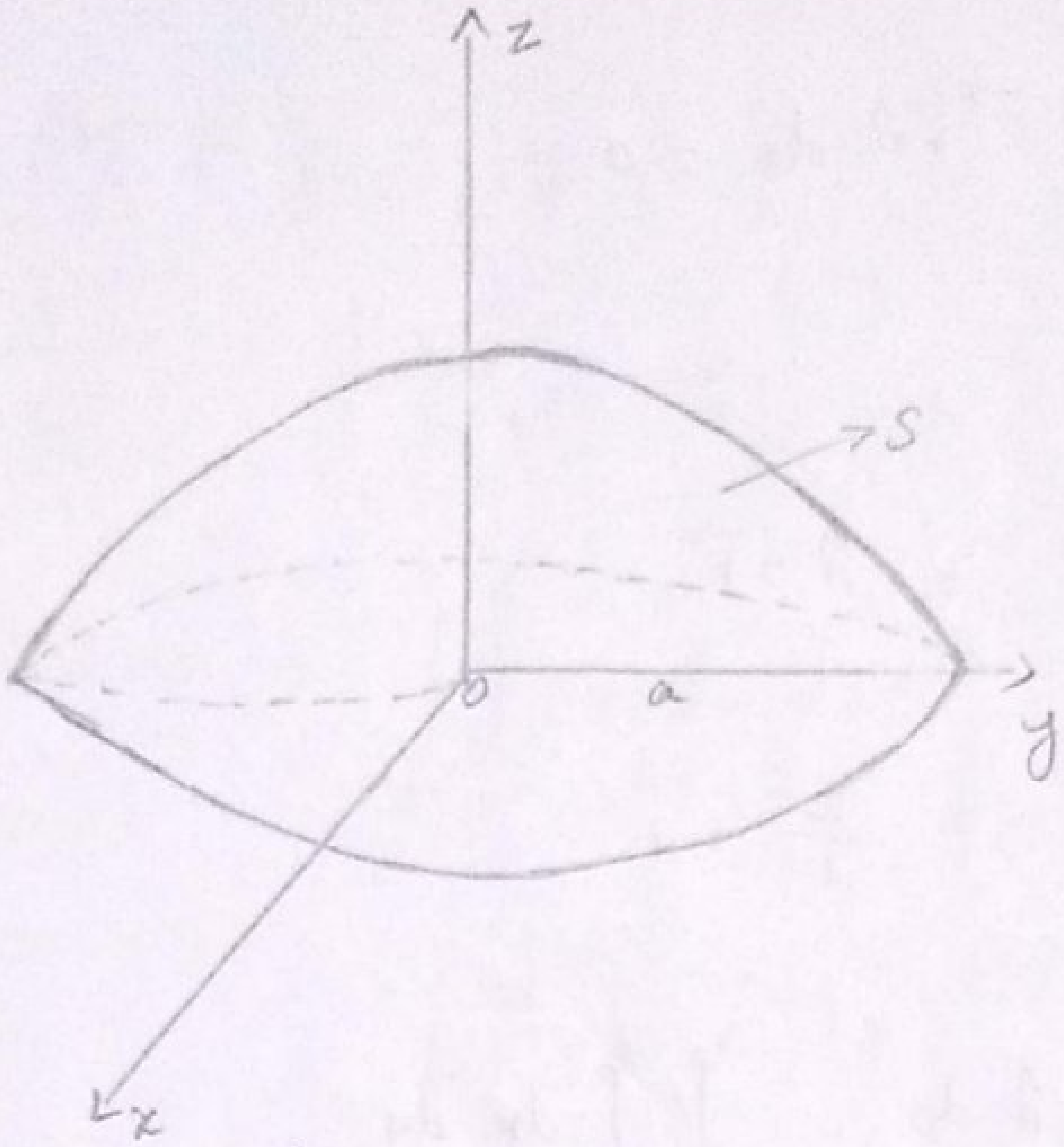
$$= 2$$

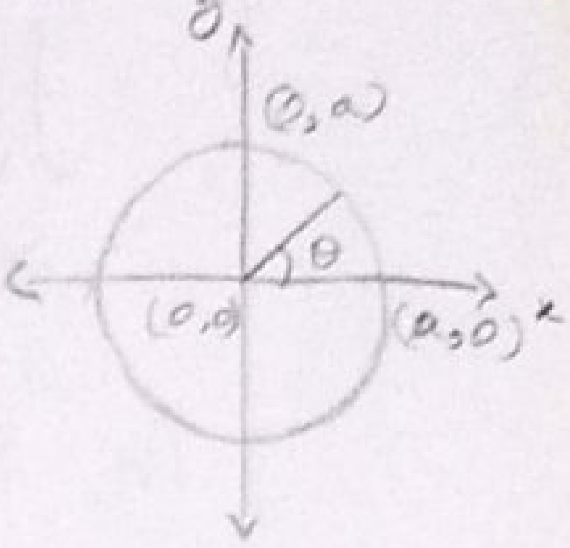
12/10/2020

Problem - 16

Evaluate $\iint_S \vec{A} \cdot \hat{n}\, ds$. if $\vec{A} = x\vec{i} + y\vec{j} - 2z\vec{k}$ and S is the surface of the sphere $x^2 + y^2 + z^2 = a^2$ above the xy plane.

Solution:





$$\phi = x^2+y^2+z^2-a^2$$

$$\nabla\phi = 2x\vec{i} + 2y\vec{j} + 2z\vec{k}$$

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{\cancel{2x\vec{i} + 2y\vec{j} + 2z\vec{k}}}{\cancel{\sqrt{4+4+4}}}$$

$$= \frac{2x\vec{i} + 2y\vec{j} + 2z\vec{k}}{\sqrt{4x^2+4y^2+4z^2}} = \frac{2x\vec{i} + 2y\vec{j} + 2z\vec{k}}{2\sqrt{x^2+y^2+z^2}}$$

$$= \frac{x\vec{i} + y\vec{j} + z\vec{k}}{\sqrt{x^2+y^2+z^2}}$$

$$= \frac{x\vec{i} + y\vec{j} + z\vec{k}}{\sqrt{a^2}}$$

$$\hat{n} = \frac{1}{a}(x\vec{i} + y\vec{j} + z\vec{k})$$

Let R be the projection of S on xoy plane.

$$\vec{A}\cdot\hat{n} = \frac{1}{a}(x^2+y^2-2z^2)$$

$$ds = \frac{dx\,dy}{|\hat{n}\cdot\vec{k}|} = \frac{dx\,dy}{z/a}$$

$$(\vec{A}\cdot\hat{n})\,ds = \frac{\frac{1}{a}(x^2+y^2-2z^2)}{z/a}\,dx\,dy$$

$$= \frac{x^2+y^2-2z^2}{z}\,dx\,dy$$

$$= \frac{x^2+y^2-2(a^2-x^2-y^2)}{\sqrt{a^2-x^2-y^2}}\,dx\,dy$$

$$= \frac{3x^2+3y^2-2a^2}{\sqrt{a^2-x^2-y^2}}\,dx\,dy$$

$$\iint_S \vec{A}\cdot\hat{n}\,ds = \iint_R \frac{3x^2+3y^2-2a^2}{\sqrt{a^2-x^2-y^2}}\,dx\,dy$$

Put $x = r\cos\theta$, $y = r\sin\theta$

Then $dxdy = rdrd\theta$

$$\iint_S \vec{A}\cdot\hat{n}\,ds = \int_0^a\int_0^{2\pi} \frac{3r^2-2a^2}{\sqrt{a^2-r^2}}\,rdr\,d\theta$$

$$= \int_0^a \frac{3r^2-2a^2}{\sqrt{a^2-r^2}}\,rdr\,[\theta]_0^{2\pi}$$

$$= \int_0^a \frac{3r^2-2a^2}{\sqrt{a^2-r^2}}\,rdr\,(2\pi)$$

$$= 2\pi\int_0^a \frac{3r^2-2a^2}{\sqrt{a^2-r^2}}\,r\,dr$$

Put $a^2-r^2 = t^2$

$-2r\,dr = 2t\,dt$

$r\,dr = -t\,dt$

$r=0 \Rightarrow t = a$

$r = a \Rightarrow t=0$

$$\iint_S \vec{A}\cdot\hat{n}\,ds = 2\pi\int_a^0 \frac{3(a^2-t^2)-2a^2}{\sqrt{t^2}}\,t\,dt$$

$$= 2\pi\int_a^0 \frac{3a^2-3t^2-2a^2}{t}\,t\,dt$$

$$= 2\pi\int_a^0 a^2-3t^2\,dt$$

$$= 2\pi\left[a^2t - 3\frac{t^3}{3}\right]_a^0$$

$$= 2\pi\left[a^2t - t^3\right]_a^0$$

$$= 2\pi\left[0 - a^3+a^3\right]$$

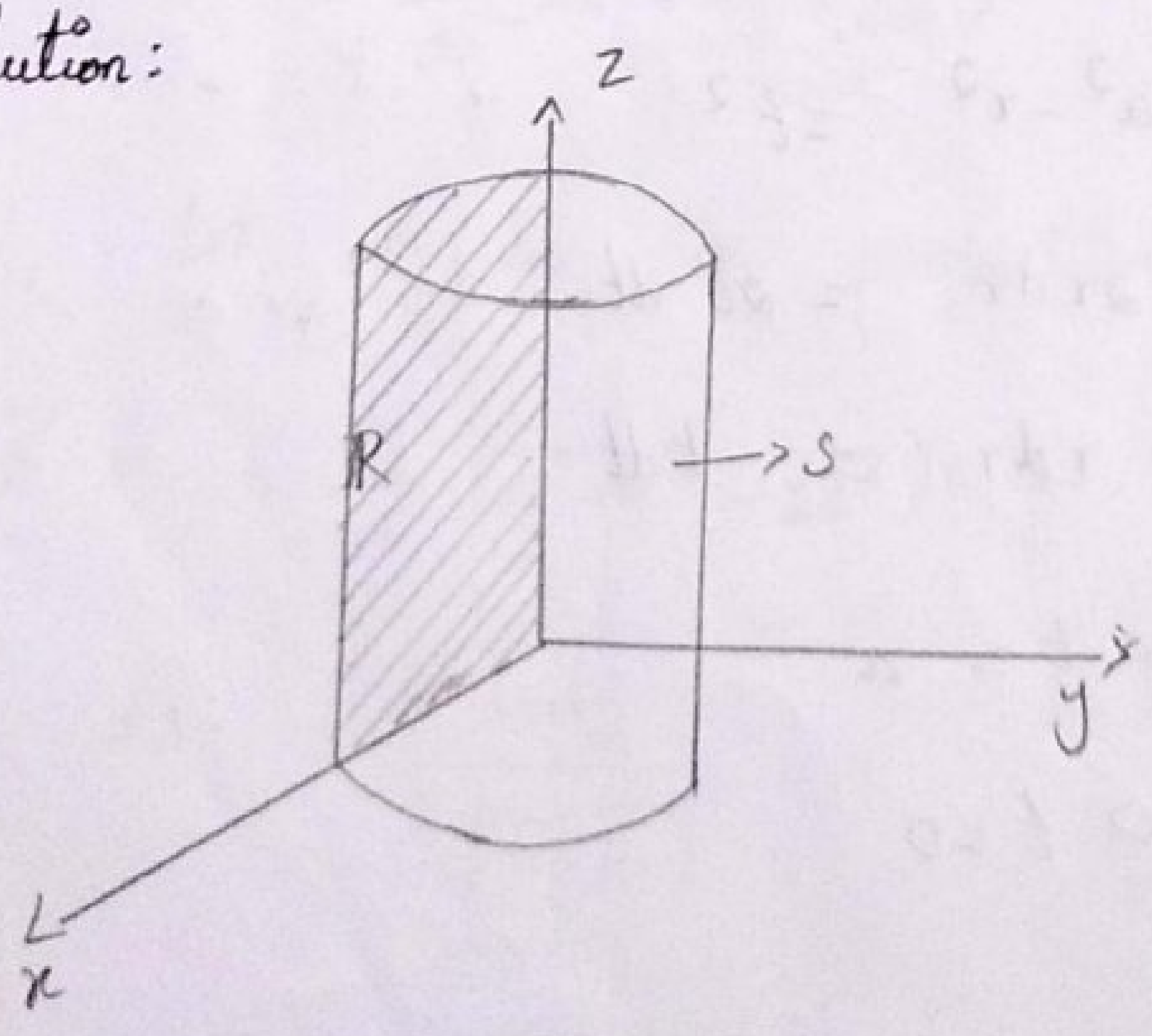
$$= 2\pi(0)$$

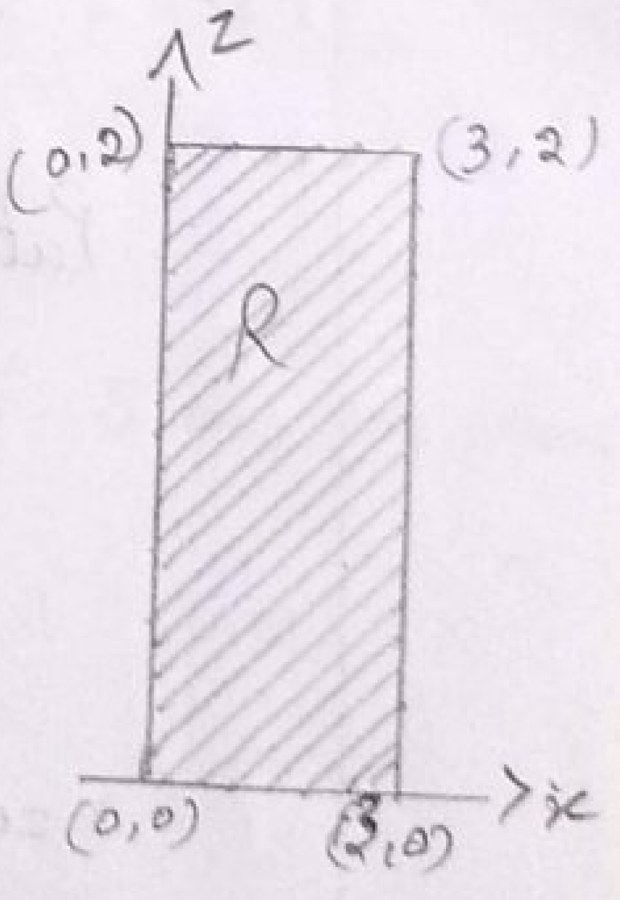
$$= 0$$

Problem-17

Evaluate $\iint_S \vec{A}\cdot\hat{n}\,ds$ if $\vec{A}=yz\vec{i}+2y^2\vec{j}+xz\vec{k}$ and S in the surface of the cylinder $x^2+y^2=9$ contained in the first octant between planes $z=0$ and $z=2$.

Solution:





Let R be the projection of S on the xoz plane.

In R,

x various from 0 to 3

z various from 0 to 2

$$\phi = x^2 + y^2 - 9$$

$$\nabla\phi = 2x\vec{i} + 2y\vec{j}$$

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2x\vec{i} + 2y\vec{j}}{\sqrt{4x^2+4y^2}} = \frac{2(x\vec{i} + y\vec{j})}{2\sqrt{x^2+y^2}}$$

$$= \frac{x\vec{i} + y\vec{j}}{\sqrt{9}} = \frac{x\vec{i} + y\vec{j}}{3}$$

$$ds = \frac{dx\,dz}{|\hat{n}.\vec{j})} = \frac{dx\,\cancel{dy}\,dz}{y/3}$$

$$\frac{\vec{A}\cdot\hat{n}}{|\hat{n}.\vec{j}|}\,dx = \frac{\frac{1}{3}[xyz + 2y^3)}{y/3}\,dx\,\cancel{dy}\,dz$$

$$= \frac{xyz + 2y^3}{y}\,dx\,\cancel{dy}\,dz$$

$$= \frac{y(xz + 2y^2)}{y}\,dx\,dz$$

$$= (xz + 2y^2)\,dx\,dz$$

$$= (xz + 2(9 - x^2))\,dx\,dz$$

$$= (xz + 18 - 2x^2)\,dx\,dz$$

$$\therefore \iint_S \vec{F}.\hat{n}\,ds = \int_0^3\int_0^2 (xz + 18 - 2x^2)\,dz\,dx$$

$$= \int_0^3 \left[x\frac{z^2}{2} + 18z - 2x^2 z\right]_0^2 dx$$

$$= \int_0^3 \left[x\frac{4}{2} + 18(2) - 4x^2\right] dx$$

$$= \int_0^3 \left[2x + 36 - 4x^2\right] dx$$

$$= \left[\frac{2x^2}{2} + 36x - 4\frac{x^3}{3}\right]_0^3$$

$$= 9 + 36(3) - 4\frac{27}{3}$$

$$= 9 + 108 - 36 \quad = 117 - 36$$

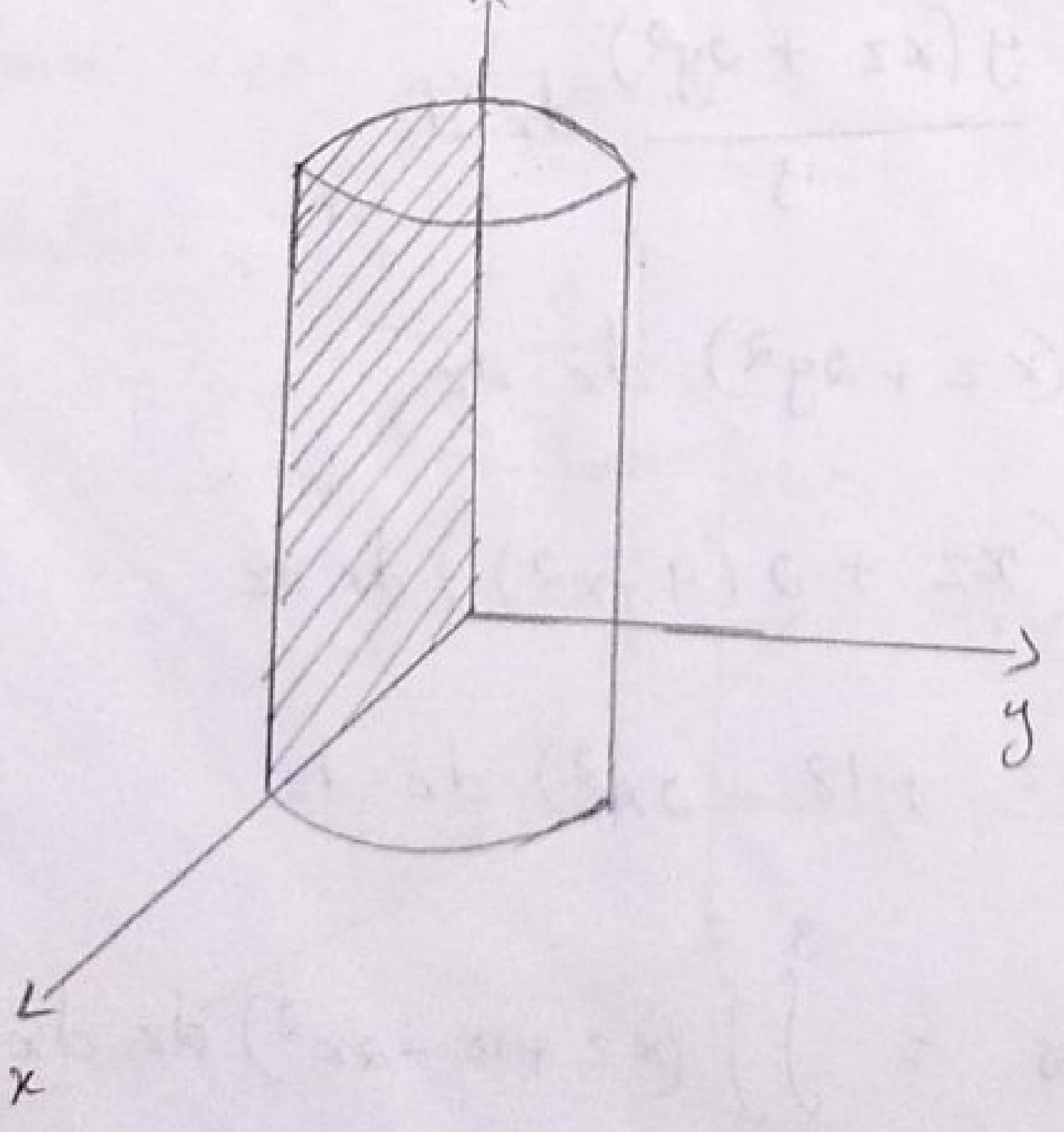
$$= 81$$

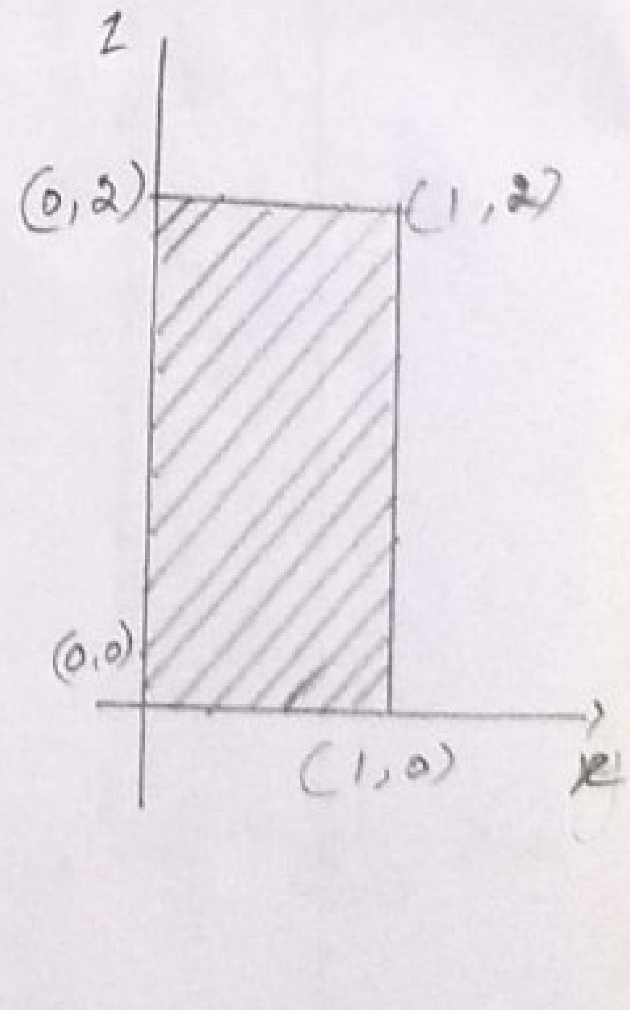
13/10/2020

Problem - 18

Evaluate $\iint_S \vec{F}\cdot\hat{n}\, ds$ where $\vec{F} = z\vec{i} + x\vec{j} - y^2 z\vec{k}$ and S is the surface of the cylinder $x^2+y^2=1$ included in the first octant between the planes $z=0$ and $z=2$.

Solution :





$$\phi = x^2+y^2-1$$

$$\nabla\phi = 2x\vec{i} + 2y\vec{j}$$

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2x\vec{i}+2y\vec{j}}{\sqrt{4x^2+4y^2}} = \frac{2(x\vec{i}+y\vec{j})}{2\sqrt{x^2+y^2}}$$

$$= \frac{x\vec{i}+y\vec{j}}{\sqrt{1}} = x\vec{i}+y\vec{j}$$

Let R be the projection of S on xz plane.

In R x various from 0 to 1

z various from 0 to 2

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iint_R \vec{F}\cdot\hat{n}\,\frac{dx\,dz}{|\hat{n}\cdot\vec{j}|}$$

$$\vec{F}\cdot\hat{n} = xz + xy$$

$$|\hat{n}\cdot\vec{j}| = y$$

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \int_0^1\int_0^2 \frac{xz+xy}{y}\,dx\,dz$$

$$= \int_0^1\int_0^2 \left(\frac{xz}{y}+x\right)dx\,dz$$

$$= \int_0^1\int_0^2 \left[\frac{xz}{\sqrt{1-x^2}}+x\right]dx\,dz$$

$$= \int_0^1 \left[\frac{x}{\sqrt{1-x^2}}\,\frac{z^2}{2}+xz\right]_0^2 dx$$

$$= \int_0^1 \left[\frac{2x}{\sqrt{1-x^2}}+2x\right]dx$$

$$= \left[-2\sqrt{1-x^2}+2\frac{x^2}{2}\right]_0^1$$

$= (-2\sqrt{1-1} + 1) - (-2\sqrt{1-0} + 0)$

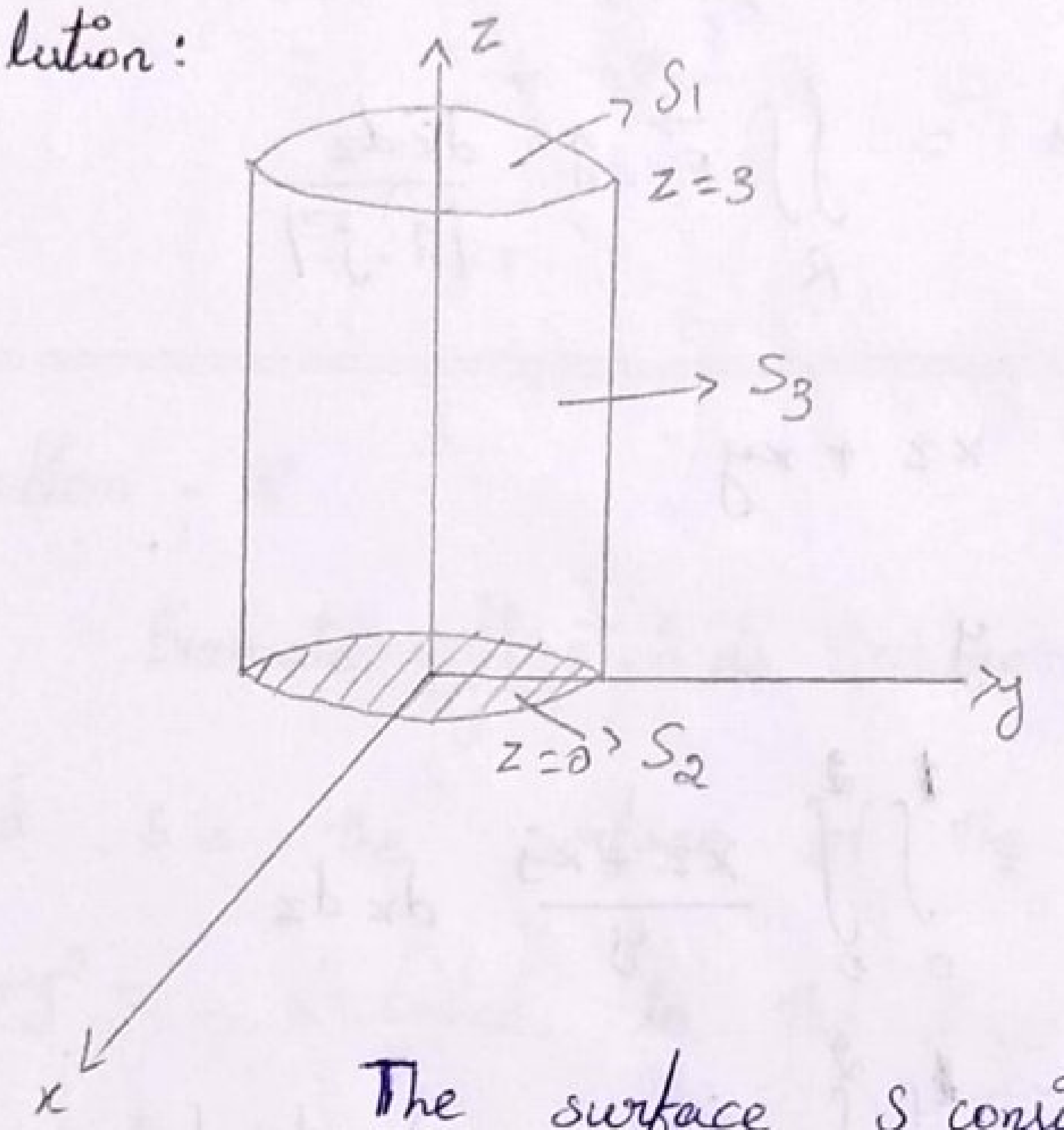
$= 0+1+2$

$= 3$

Problem - 19

Evaluate $\iint_S \vec{F}\cdot\hat{n}\,ds$ where $\vec{F} = 4x\vec{i} - 2y^2\vec{j} + z^2\vec{k}$ and S is the surface of the region $x^2+y^2=4$, $z=0$ and $z=3$.

Solution:



The surface S consists of three parts. S_1 = the circle in the plane $z=3$

S_2 = the circle in the plane $z=0$

S_3 = the curved surface of the cylinder $x^2+y^2=4$

$$\therefore \iint_S \vec{F}\cdot\hat{n}\,ds = \iint_{S_1} + \iint_{S_2} + \iint_{S_3} (\vec{F}\cdot\hat{n}\,ds) \rightarrow ①$$

On S_1

$z=3$, $\hat{n}=\vec{k}$

$\vec{F}\cdot\hat{n} = z^2 = 9$

$$\iint_{S_1} \vec{F}\cdot\hat{n}\, ds = \iint_R \vec{F}\cdot\hat{n} \frac{dx\,dy}{|\hat{n}\cdot\vec{k}|}$$

$$= \iint_R 9 \frac{dx\,dy}{1}$$

$$= 9\iint_R dx\,dy$$

$= 9 \times$ circle area

$= 9 \times \pi r^2$

$= 9 \times \pi(2)^2$

$= 36\pi$

On S_2

$z = 0$, $\hat{n} = -\vec{k}$

$\vec{F}\cdot\hat{n} = -z^2 = 0$

$$\iint_{S_2} \vec{F}\cdot\hat{n}\, ds = 0$$

On S_3

~~z~~ $= x^2+y^2 = 4$

$x = 2\cos\theta \qquad y = 2\sin\theta$

$ds = 2\,d\theta\, dz$

$\phi = x^2+y^2-4$

$\nabla\phi = 2x\vec{i} + 2y\vec{j}$

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2x\vec{i}+2y\vec{j}}{\sqrt{4x^2+4y^2}} = \frac{2(x\vec{i}+y\vec{j})}{2\sqrt{x^2+y^2}}$$

$$= \frac{x\vec{i}+y\vec{j}}{\sqrt{4}} = \frac{x}{2}\vec{i} + \frac{y}{2}\vec{j}$$

$$\vec{F}\cdot\hat{n} = \frac{4x^2}{2} + \left(-\frac{2y^3}{2}\right)$$

$$= 2x^2 - y^3$$

$$\iint_{S_3} \vec{F}\cdot\hat{n}\, ds = \iint_{S_3} (2x^2-y^3)\ 2\ d\theta\, dz$$

~~$$= 2\int_0^3\int_0^{2\pi} (8\cos^2\theta - 8\sin^3\theta)\, d\theta\, dz$$~~

~~$$= 8\times 2 \int_0^3\int_0^{2\pi} \cos^2\theta .$$~~

$$= 2\int_0^3\int_0^{2\pi} (8\cos^2\theta - 8\sin^3\theta)\, d\theta\, dz$$

$$= 2\times 8 \int_0^3\int_0^{2\pi} (\cos^2\theta - \sin^3\theta)\, d\theta\, dz$$

$$= 16 \int_0^{2\pi} (\cos^2\theta - \sin^3\theta) d\theta \left[z\right]_0^3$$

$$= 16 \int_0^{2\pi} (\cos^2\theta - \sin^3\theta)\ (3)\ d\theta$$

$$= 16\times 3 \int_0^{2\pi} \left[\frac{1+\cos 2\theta}{2} - \frac{1}{4}\left(3\sin\theta - \sin 3\theta\right)\right] d\theta$$

$$= 48 \int_0^{2\pi} \left[\frac{1}{2} + \frac{\cos 2\theta}{2} - \frac{3}{4}\sin\theta + \frac{1}{4}\sin 3\theta\right] d\theta$$

$$= 48 \left[\frac{1}{2}\theta + \frac{\sin 2\theta}{2\times 2} - \frac{3}{4}(-\cos\theta) + \frac{1}{4}\left(-\frac{\cos 3\theta}{3}\right)\right]_0^{2\pi}$$

$$= 48 \left[\frac{1}{2}\theta + \frac{\sin 2\theta}{4} + \frac{3\cos\theta}{4} - \frac{\cos 3\theta}{12}\right]_0^{2\pi}$$

$$=48\left\{\left[\pi+0+\frac{3}{4}-\frac{1}{12}\right]-\left[0+0+\frac{3}{4}-\frac{1}{12}\right]\right\}$$

$$=48\left[\pi\right]$$

$$=48\pi$$

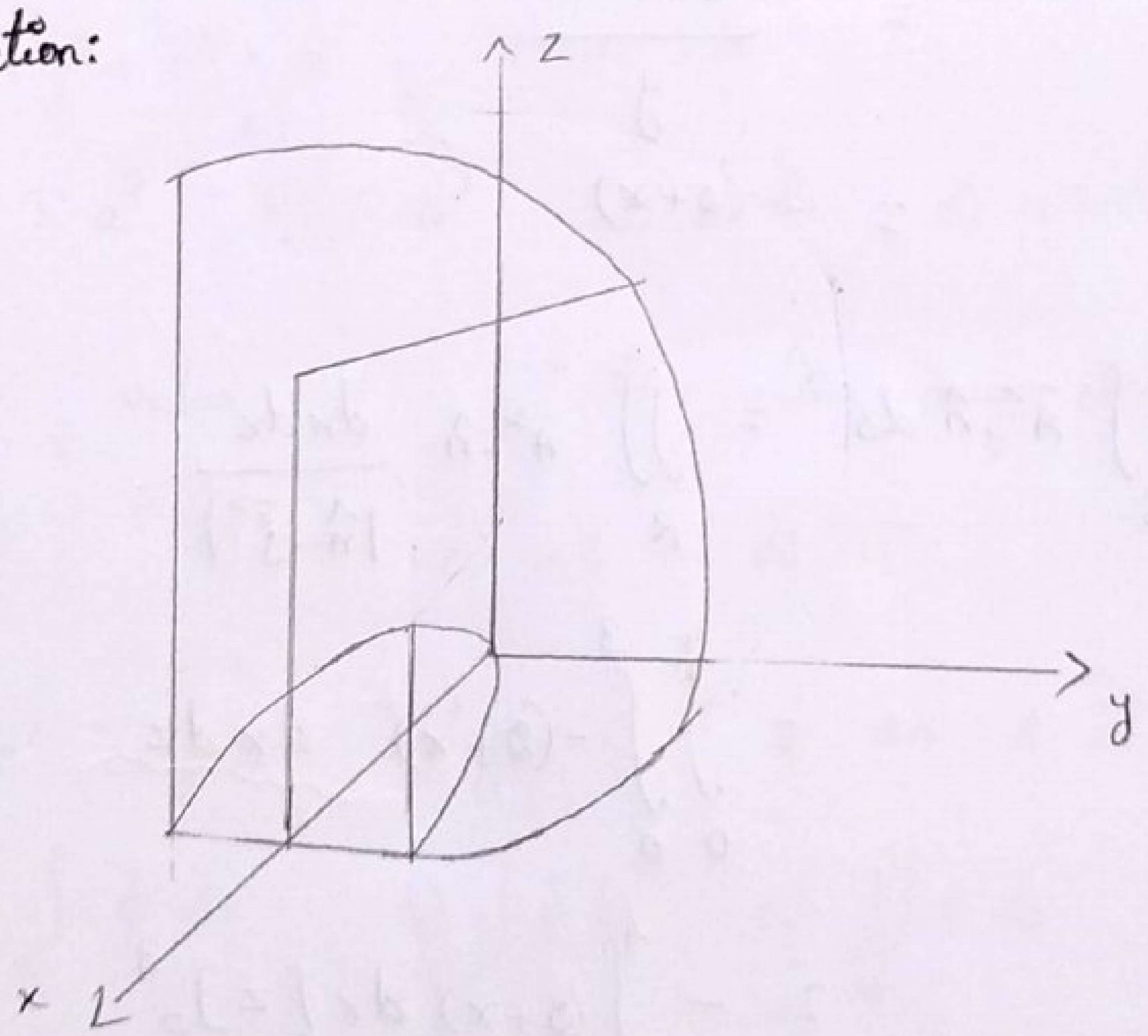
$$\iint_S \vec{F}\cdot\hat{n}\,ds = 36\pi+0+48\pi$$

$$=84\pi$$

16/10/2020 Problem - 20

Evaluate $\iint_S \vec{A}\cdot\hat{n}\,ds$ if $\vec{A}=y\vec{i}-x\vec{j}+z\vec{k}$ and S is the surface of the parabolic cylinder $y^2-4x=0$ in the first actant in the planes $x=4$ and $z=3$.

Solution:



$$\phi = y^2-4x$$

$$\nabla\phi = -4\vec{i}+2y\vec{j}$$

$$\hat{n}=\frac{\nabla\phi}{|\nabla\phi|}=\frac{-4\vec{i}+2y\vec{j}}{\sqrt{16+4y^2}}=\frac{-4\vec{i}+2y\vec{j}}{\sqrt{4(4+y^2)}}$$

$$= \frac{2(-2\vec{i} + y\vec{j})}{2\sqrt{4+y^2}}$$

$$= \frac{-2\vec{i} + y\vec{j}}{\sqrt{4+y^2}}$$

Let R be the projection of S on xz plane.

In R, x varies from 0 to 4

z various from 0 to 3.

$$\hat{n}\cdot\vec{j} = \frac{y}{\sqrt{4+y^2}}$$

$$\frac{\vec{A}\cdot\hat{n}}{|\hat{n}\cdot\vec{j}|} = \frac{-2y-xy}{\sqrt{4+y^2}} \Big/ \frac{y}{\sqrt{4+y^2}}$$

$$= \frac{-y(2+x)}{y}$$

$$= -(2+x)$$

$$\therefore \iint_S \vec{A}\cdot\hat{n}\,ds = \iint_R \vec{A}\cdot\hat{n}\,\frac{dx\,dz}{|\hat{n}\cdot\vec{j})}$$

$$= \int_0^4\int_0^3 -(2+x)\,dx\,dz$$

$$= -\int_0^4 (2+x)\,dx\,[z]_0^3$$

$$= -\int_0^4 (2+x)(3)\,dx$$

$$= -3\left[2x + \frac{x^2}{2}\right]_0^4$$

$$= -3\left[8 + \frac{16}{2}\right]$$

$= -3\left[\frac{16+16}{2}\right]$

$= -3\left[\frac{32}{2}\right]$

$= -3(16)$

$= -48$

Part - A

1) If $\vec{f} = 3xy\,\vec{i} - y^2\vec{j}$ and $x = t$, $y = 2t^2$ then $\vec{F}\cdot\vec{dr}$ is

a) $2t^3\,dt - 5t^2\,dt$ b) $t^3\,dt - 16t^5\,dt$

c) $6t^3\,dt + 16t^4\,dt$ d) $2t^3\,dt + 5t^4\,dt$

2) The surface area of the semi sphere $x^2+y^2+z^2 = a^2$, $z \geq 0$ is

a) πa^2 b) $\pi 2a^2$ c) $3\pi a^2$ d) $4\pi a^2$

3) The value of $\int \vec{r}\cdot\vec{dr}$ along any closed curve is

a) 0 b) 2π c) $-\pi$ d) π

4) The surface integral of ϕ on S is denoted by

a) $\int_C \phi\, dx$ b) $\int_S \phi\, ds$ c) $\iint_S \phi\, ds$ d) $\iint_C \phi\, dr$.

5) The line integral $\int_C \vec{F}\cdot\vec{dr}$ is

a) $\int_C (F_1\,dx - F_2\,dy - F_3\,dz)$

b) $\int_C (F_1\,dx + F_2\,dy + F_3\,dz)$

c) $\int_C (F_1 + F_2 + F_3)$ d) none.

6) The surface integral is a

a) vector b) constant c) scalar d) point function

7) If $\vec{F}$ is a conservative vector field and

$\vec{F} = \nabla\phi$, then ϕ is called

a) Potential b) scalar potential

c) scalar d) derivative

17/10/2020

Unit - IV & V

Volume Integral:

The volume integral $\vec{f}$ over the volume V is denoted by $\iiint_V \vec{f} \cdot \vec{dv}$ (or) $\iiint_V \nabla \cdot \vec{f}\, dv$.

If $\vec{f} = f_1 \vec{i} + f_2 \vec{j} + f_3 \vec{k}$, then

$$\iiint_V \vec{f} \cdot \vec{dv} = \vec{i} \iiint_V f_1\, dv + \vec{j} \iiint_V f_2\, dv + \vec{k} \iiint_V f_3\, dv$$

---

1) Evaluate $\iiint_V \nabla \cdot \vec{F}\, dv$. If $\vec{F} = x^2 \vec{i} + y^2 \vec{j} + z^2 \vec{k}$ and V is the volume of the region enclosed by cube $0 \leq x, y, z \leq 1$.

Solution :

$$\vec{F} = x^2 \vec{i} + y^2 \vec{j} + z^2 \vec{k}$$

$$\nabla \cdot \vec{F} = 2x + 2y + 2z = 2(x+y+z)$$

$$\therefore \iiint_V \nabla \cdot \vec{F}\, dv = 2 \int_0^1 \int_0^1 \int_0^1 (x+y+z)\, dx\, dy\, dz$$

$$= 2 \int_0^1 \int_0^1 \left[\frac{x^2}{2} + xy + zx \right]_0^1 dy\, dz$$

$$= 2 \int_0^1 \int_0^1 \left[\frac{1}{2} + y + z \right] dy\, dz$$

$$= 2 \int_0^1 \left[\frac{1}{2} y + \frac{y^2}{2} + yz \right]_0^1 dz$$

$$= 2 \int_0^1 \left[\frac{1}{2} + \frac{1}{2} + z \right] dz$$

$$= 2\left[\frac{1}{2}z + \frac{1}{2}z + \frac{z^2}{2}\right]_0^1$$

$$= 2\left[\frac{1}{2} + \frac{1}{2} + \frac{1}{2}\right]$$

$$= 2\left(\frac{3}{2}\right)$$

$$= 3$$

Gauss Divergence Theorem (GDT)

If V is the volume of a closed surface S and $\vec{A}$, a vector point function with continuous derivatives in V, then

$$\iint_S \vec{A}\cdot\hat{n}\,ds = \iiint_V \nabla\cdot\vec{A}\,dv$$

Note:

Scalar form of GDT

$$\iint_S \left(A_1\,dydz + A_2\,dzdx + A_3\,dxdy\right)$$

$$= \iiint_V \left(\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}\right) dxdydz$$

Green's Theorem in Plane:

If C is a simple closed curve in the xoy plane bounding an area R and M (x, y) and N (x, y) are continuous functions of x and y having continuous derivatives in R, then

$$\oint (M\,dx + N\,dy) = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

Stokes Theorem:

If S is bounded by a simple close curve C, then.

$$\oint \vec{A}\cdot d\vec{r} = \iint_S (\nabla \times \vec{A})\,\vec{ds} = \iint_S (\nabla \times \vec{A})\cdot \hat{n}\,ds$$

where $\vec{A}$ has continuous derivatives on S and $\hat{n}$ is the unit vector normal to S.

---

2. Find $\iint_S \vec{r}\cdot\hat{n}\,ds$ if S is the surface of the sphere $x^2+y^2+z^2=a^2$.

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Solution:

By Gauss ~~Divg~~ Divergence Theorem (GDT).

$$\iint_S \vec{r}\cdot\hat{n}\,ds = \iiint_V \nabla\cdot\vec{r}\,dv$$

$$\nabla\cdot\vec{r} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z)$$

$$= 1+1+1 = 3$$

$$\iint_S \vec{r}\cdot\hat{n}\,ds = \iiint_V 3\,dv$$

$$= 3\iiint_V dv$$

$$= 3 \times (\text{volume enclosed by } S)$$

$$= 3\times\left(\frac{4}{3}\pi a^3\right)$$

$$= 4\pi a^3$$

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3) Evaluate $\iint\limits_S (x\, dy\, dz + y\, dz\, dx + z\, dx\, dy)$ over the surface of the sphere $x^2+y^2+z^2=a^2$

Solution:

By GDT, we have

$$\iint\limits_S (A_1\, dy\, dz + A_2\, dz\, dx + A_3\, dx\, dy)$$

$$= \iiint\limits_V \left(\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}\right) dx\, dy\, dz$$

$$\therefore \iint\limits_S (x\, dy\, dz + y\, dz\, dx + z\, dx\, dy)$$

$$= \iiint\limits_V \left(\frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z}\right) dx\, dy\, dz$$

$$= \iiint\limits_V (1+1+1)\, dx\, dy\, dz$$

$$= 3 \iiint\limits_V dx\, dy\, dz$$

$$= 3 \times (\text{volume enclosed by } S)$$

$$= 3 \times \frac{4}{3}\pi a^3$$

$$= 4\pi a^3$$

---

4. Evaluate $\iint\limits_S \vec{A} \cdot \vec{ds}$ if $\vec{A} = 2xy\,\vec{i} + xz^2\vec{j} + xz\,\vec{k}$ and S is the surface of the parallapiped formed by the planes $x=0$, $x=2$, $y=0$, $y=1$, $z=0$ and $z=3$.

Solution:

By ~~Gauss~~ Gauss Divergence Theorem (GDT)

$$\iint_S \vec{A}\cdot\vec{ds} = \iiint_V \nabla\cdot\vec{A}\, dv$$

$$\nabla\cdot\vec{A} = \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}\right)\cdot\left(2xy\vec{i} + xz^2\vec{j} + xz\vec{k}\right)$$

$$= 2y + x$$

$$\therefore \iint_S \vec{A}\cdot\vec{ds} = \int_0^3\int_0^1\int_0^2 (2y+x)\, dx\, dy\, dz$$

$$= \int_0^3\int_0^1 \left[2yx + \frac{x^2}{2}\right]_0^2 dy\, dz$$

$$= \int_0^3\int_0^1 \left[4y + 2\right] dy\, dz$$

$$= \int_0^3 \left[\frac{4y^2}{2} + 2y\right]_0^1 dz$$

$$= \int_0^3 \left[2 + 2\right] dz$$

$$= \int_0^3 4\, dz$$

$$= \left[4z\right]_0^3$$

$$= 12$$

---

5) Evaluate $\iint_S \vec{A}\cdot\vec{ds}$ if $\vec{A} = 4xz\vec{i} - y^2\vec{j} + yz\vec{k}$ and S is the surface of the cube bounded by $x=0$, $x=1$, $y=0$, $y=1$, $z=0$, $z=1$

Solution:

By Gauss Divergence Theorem (GDT), we have,

$$\iint_S \vec{A} \cdot \vec{ds} = \iiint_V \nabla \cdot \vec{A}\, dv$$

$$\nabla \cdot \vec{A} = \left(\vec{i} \frac{\partial}{\partial x} + \vec{j} \frac{\partial}{\partial y} + \vec{k} \frac{\partial}{\partial z} \right) \cdot \left(4xz\vec{i} - y^2\vec{j} + yz\vec{k} \right)$$

$$= 4z - 2y + y$$

$$= 4z - y$$

$$\therefore \iint_S \vec{A} \cdot \vec{ds} = \int_0^1\int_0^1\int_0^1 (4z - y)\, dx\, dy\, dz$$

$$= \int_0^1\int_0^1 (4xz - xy)_0^1\, dy\, dz$$

$$= \int_0^1\int_0^1 (4z - y)\, dy\, dz$$

$$= \int_0^1 \left[4zy - \frac{y^2}{2} \right]_0^1 dz$$

$$= \int_0^1 \left[4z - \frac{1}{2} \right] dz$$

$$= \left[\frac{4z^2}{2} - \frac{1}{2}z \right]_0^1$$

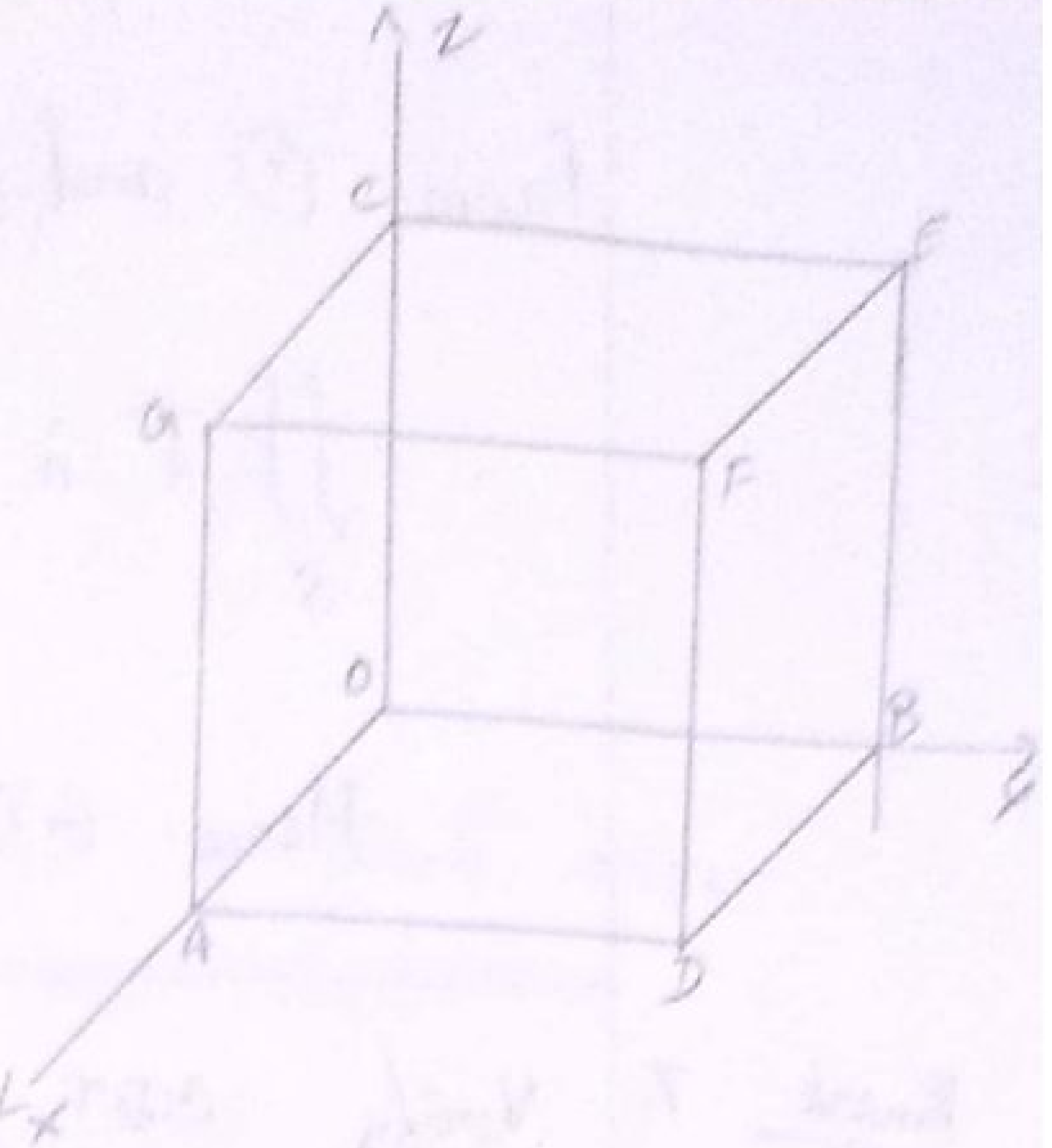
$$= \cancel{\frac{4}{2}} - \cancel{\frac{1}{2}}\ \frac{4}{2} - \frac{1}{2}$$

$$= \frac{3}{2}$$

20/10/2020
8m

6. Verify for $\vec{A} = 4xz\vec{i} - y^2\vec{j} + yz\vec{k}$ and S is the surface of the cube bounded by $x=0,\ x=1,\ y=0,\ y=1,\ z=0,\ z=1$

Solution:



$$\iint_S \vec{A} \cdot \hat{n}\, ds = \iiint_V \nabla \cdot \vec{A}\, dv$$

we have,

$$\iiint_V \nabla \cdot \vec{A}\, dv = 3/2 \longrightarrow ①$$

To find, $\iint_S \vec{A} \cdot \hat{n}\, ds$

$$\iint_S \vec{A} \cdot \hat{n}\, ds = \iint_{O.ADB} + \iint_{CGFE} + \iint_{OBEC} + \iint_{ADFG} + \iint_{OACG} + \iint_{DBEF} (\vec{A} \cdot \hat{n}\, ds)$$

$$= \iint_{\substack{z=0 \\ \hat{n} = -\vec{k}}} \vec{A} \cdot \hat{n}\, ds + \iint_{\substack{z=1 \\ \hat{n} = \vec{k}}} \vec{A} \cdot \hat{n}\, ds + \iint_{\substack{x=0 \\ \hat{n} = -\vec{i}}} \vec{A} \cdot \hat{n}\, ds +$$

$$\iint_{\substack{x=1 \\ \hat{n} = \vec{i}}} \vec{A} \cdot \hat{n}\, ds + \iint_{\substack{y=0 \\ \hat{n} = -\vec{j}}} \vec{A} \cdot \hat{n}\, ds + \iint_{\substack{y=1 \\ \hat{n} = \vec{j}}} \vec{A} \cdot \hat{n}\, ds$$

$$= \int_0^1\int_0^1 0 + \int_0^1\int_0^1 y\, dx\, dy + \int_0^1\int_0^1 0 + \int_0^1\int_0^1 4z\, dy\, dz +$$

$$\int_0^1\int_0^1 0 - \int_0^1\int_0^1 dx\, dz$$

$$= 0 + [x]_0^1 \left[\frac{y^2}{2}\right]_0^1 + 0 + \left[\frac{4z}{2}\right]_0^1 [y]_0^1 + 0$$

$$- [x]_0^1 [z]_0^1$$

$$= (1)(1/2) + (2)(1) - (1)(1)$$

$$= \frac{1}{2} + 2 - 1 \quad = \frac{1}{2} + 1$$

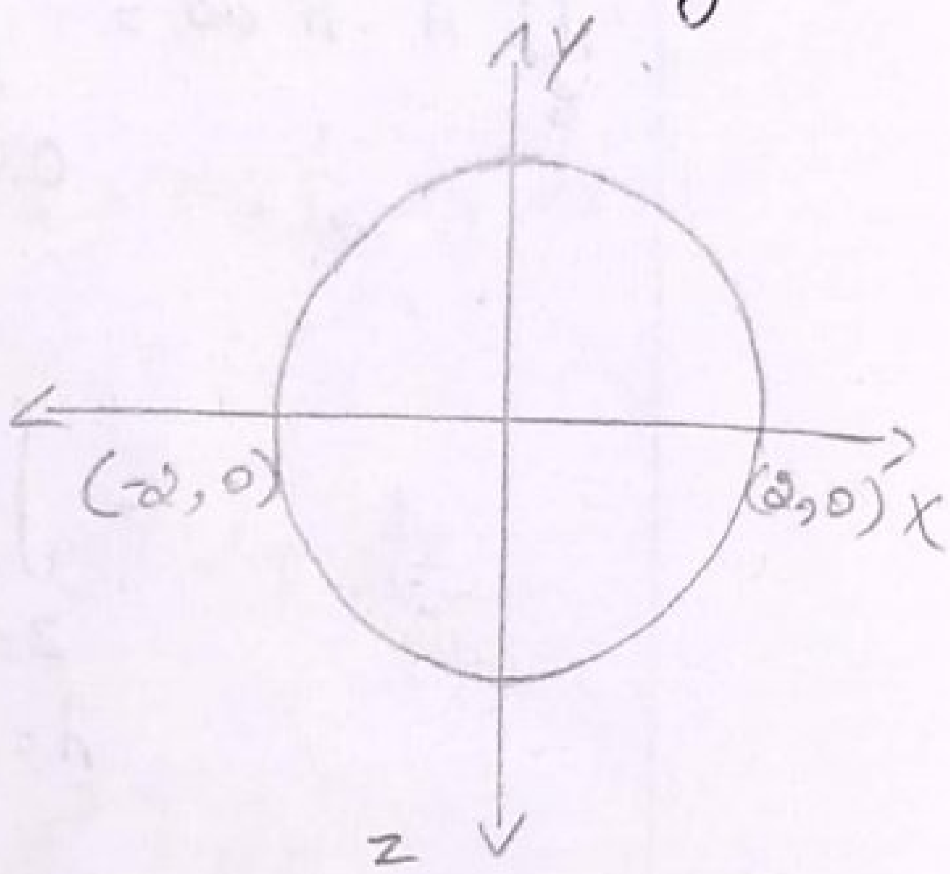
$$\iint_S \vec{A} \cdot \hat{n}\, ds = 3/2 \longrightarrow ②$$

From ① and ②, we get

$$\iint_S \vec{A}\cdot\hat{n}\,ds = \iiint_V \cancel{A}\,\nabla\cdot\vec{A}\,dv$$

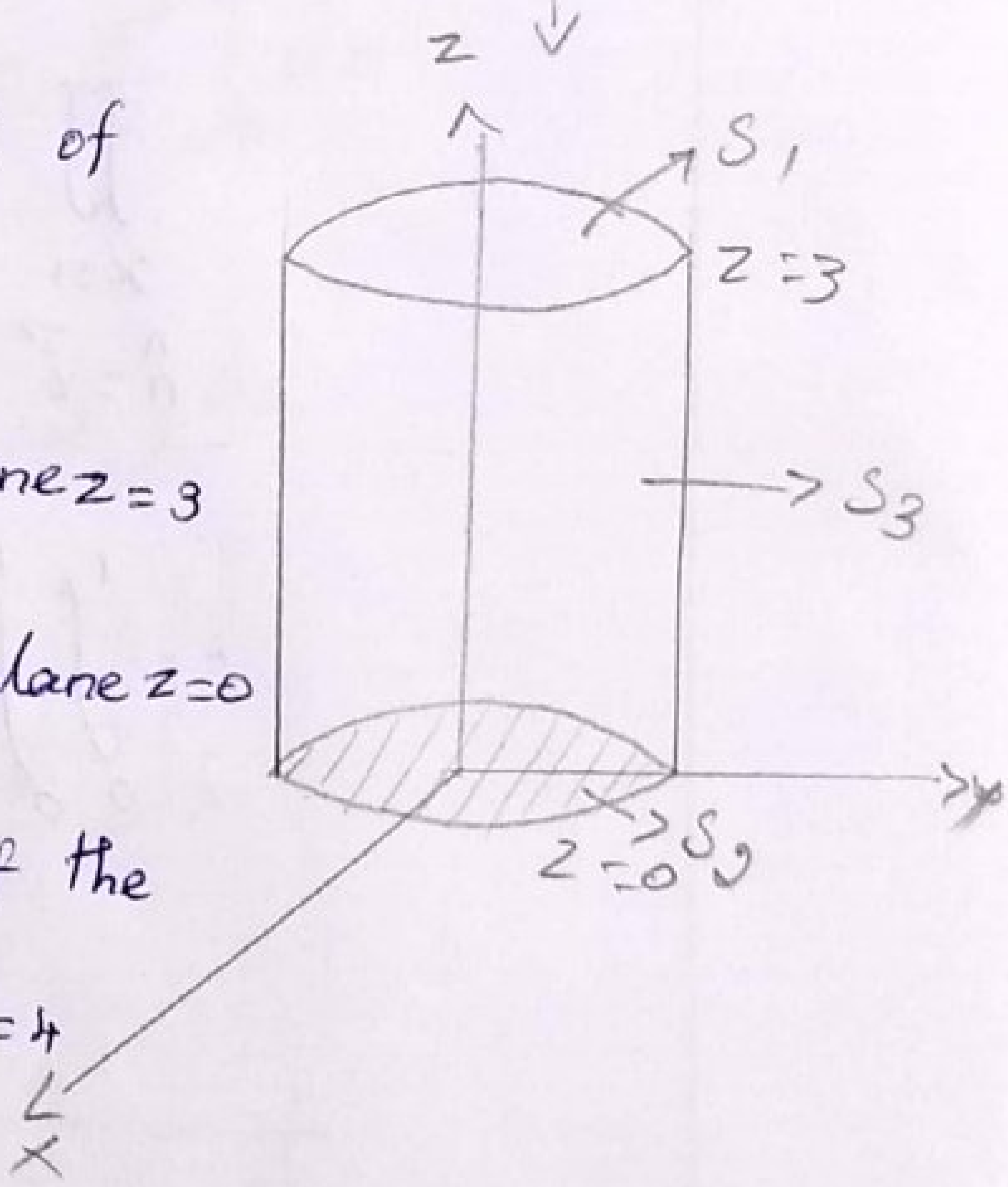
Hence GDT is verified.

8mark 7. Verify GDT for $\vec{A} = 4x\vec{i} - 2y^2\vec{j} + z^2\vec{k}$ and S is the surface of the region bounded by $x^2+y^2=4$, $z=0$, $z=3$.



Solution:

$$\iint_S \vec{A}\cdot\hat{n}\,ds = \iiint_V \nabla\cdot\vec{A}\,dv$$



The surface S consists of three parts.

S_1 = The circle in the plane $z=3$

S_2 = The circle in the plane $z=0$

S_3 = The curved surface of the cylinder $x^2+y^2=4$

$$\therefore \iint_S \vec{A}\cdot\hat{n}\,ds = \iint_{S_1} + \iint_{S_2} + \iint_{S_3} (\vec{A}\cdot\hat{n}\,ds) \rightarrow ①$$

On S_1

$z=3$, $\hat{n}=\vec{k}$

$\vec{A}\cdot\hat{n} = z^2 = 9$

$$\iint_{S_1} \vec{A}\cdot\hat{n}\, ds = \iint_{R} \vec{A}\cdot\hat{n}\,\frac{dx\,dy}{|\hat{n}\cdot\vec{k}|}$$

$$= \iint_{R} 9\,\frac{dx\,dy}{1}$$

$$= 9\iint_{R} dx\,dy = 9\times \text{circle area}$$

$$= 9\times\pi r^2 = 9\pi(2)^2 = 9\pi(4)$$

$$= 36\pi$$

On S_2

$z=0,\ \hat{n}=-\vec{k}$

$\vec{A}\cdot\hat{n} = -z^2 = 0$

$$\iint_{S_2} \vec{A}\cdot\hat{n}\, ds = 0$$

On S_3

$x^2+y^2=4$

$x = 2\cos\theta \qquad y = 2\sin\theta$

$ds = 2\,d\theta\,dz$

$\phi = x^2+y^2-4$

$\nabla\phi = 2x\vec{i}+2y\vec{j}$

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2x\vec{i}+2y\vec{j}}{\sqrt{4x^2+4y^2}} = \frac{2x\vec{i}+2y\vec{j}}{\sqrt{4(x^2+y^2)}}$$

$$= \frac{2(x\vec{i}+y\vec{j})}{2\sqrt{x^2+y^2}} = \frac{x\vec{i}+y\vec{j}}{\sqrt{4}} = \frac{x}{2}\vec{i}+\frac{y}{2}\vec{j}$$

$$\vec{A}\cdot\hat{n} = \frac{4x^2}{2} + \left(\frac{-2y^3}{2}\right)$$

$$= 2x^2 - y^3$$

$$\iint_{S_3} \vec{A}\cdot\hat{n}\, ds = \iint_{S_3} (2x^2 - y^3)\; 2\, d\theta\, dz$$

$$= 2\int_0^3\int_0^{2\pi} (8\cos^2\theta - 8\cos^3\theta)\, d\theta\, dz$$

$$= 2\times 8 \int_0^3\int_0^{2\pi} (\cos^2\theta - \cos^3\theta)\, d\theta\, dz$$

$$= 16 \int_0^{2\pi} (\cos^2\theta - \cos^3\theta)\, [z]_0^3$$

$$= 16\times 3 \int_0^{2\pi} \left[\frac{1+\cos 2\theta}{2} - \frac{3\sin\theta - \sin 3\theta}{4}\right] d\theta$$

$$= 48 \int_0^{2\pi} \left[\frac{1}{2} + \frac{\cos 2\theta}{2} - \frac{3\sin\theta}{4} + \frac{\sin 3\theta}{4}\right] d\theta$$

$$= 48 \left[\frac{\theta}{2} + \frac{\sin 2\theta}{4} + \frac{3\cos\theta}{4} - \frac{\overset{\cos 3\theta}{\cancel{\sin 2\theta}}}{12}\right]_0^{2\pi}$$

$$= 48\left\{\left[\pi + 0 + \frac{3}{4} - \frac{1}{12}\right] - \left[0 + 0 + \frac{3}{4} - \frac{1}{12}\right]\right\}$$

$$= 48\left\{\pi + \frac{3}{4} - \frac{1}{12} - \frac{3}{4} + \frac{1}{12}\right\}$$

$$= 48\pi$$

From ①

$$\iint_S \vec{A}\cdot\hat{n}\, ds = 36\pi + 0 + 48\pi$$

$$= 84\pi \longrightarrow ②$$

$$\nabla \cdot \vec{A} = \frac{\partial}{\partial x}(4x) + \frac{\partial}{\partial y}(-2y^2) + \frac{\partial}{\partial z}(z^2)$$

$$= 4 - 4y + 2z$$

$$\iiint_V \nabla \cdot \vec{A}\, dv = \int_0^3 \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (4 - 4y + 2z)\, dx\, dy\, dz$$

$$= \int_0^3 \int_{-2}^{2} \left[\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (4 + 2z)\, dy + \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (-4y)\, dy \right] dx\, dz$$

$$= \int_0^3 \int_{-2}^{2} \left[2 \int_0^{\sqrt{4-x^2}} (4 + 2z)\, y\, dy \right] dx\, dz$$

$$= \int_0^3 \int_{-2}^{2} (4 + 2z) \sqrt{4 - x^2}\, dx\, dz$$

$$= 2 \times 2 \int_0^3 \int_0^2 (4 + 2z) \sqrt{4 - x^2}\, dx\, dz$$

$$= 4 \int_0^3 (4 + 2z) \left[\frac{x}{2} \sqrt{4 - x^2} + \frac{4}{2} \sin^{-1} \frac{x}{2} \right]_0^2 dz$$

~~$= 4 \int_0^3 (4 + 2z) \frac{\pi}{2}\, dz$~~

$$= 4 \int_0^3 (4 + 2z) \left[0 + 2\left(\frac{\pi}{2}\right) \right] dz$$

$$= 4\pi \int_0^3 (4 + 2z)\, dz$$

$$= 4\pi \left[4z + \frac{2z^2}{2} \right]_0^3$$

$$= 4\pi \left[4z + z^2 \right]_0^3$$

$$= 4\pi \left[12 + 9 \right]$$

$$= 4\pi \left[21 \right]$$

$$\iiint_V \nabla \cdot \vec{A} \, ds = 84\pi \longrightarrow ③$$

From ② and ③, we get

$$\iint_S \vec{A} \cdot \hat{n} \, ds = \iiint_V \nabla \cdot \vec{A} \, ds$$

Hence GDT is verified.

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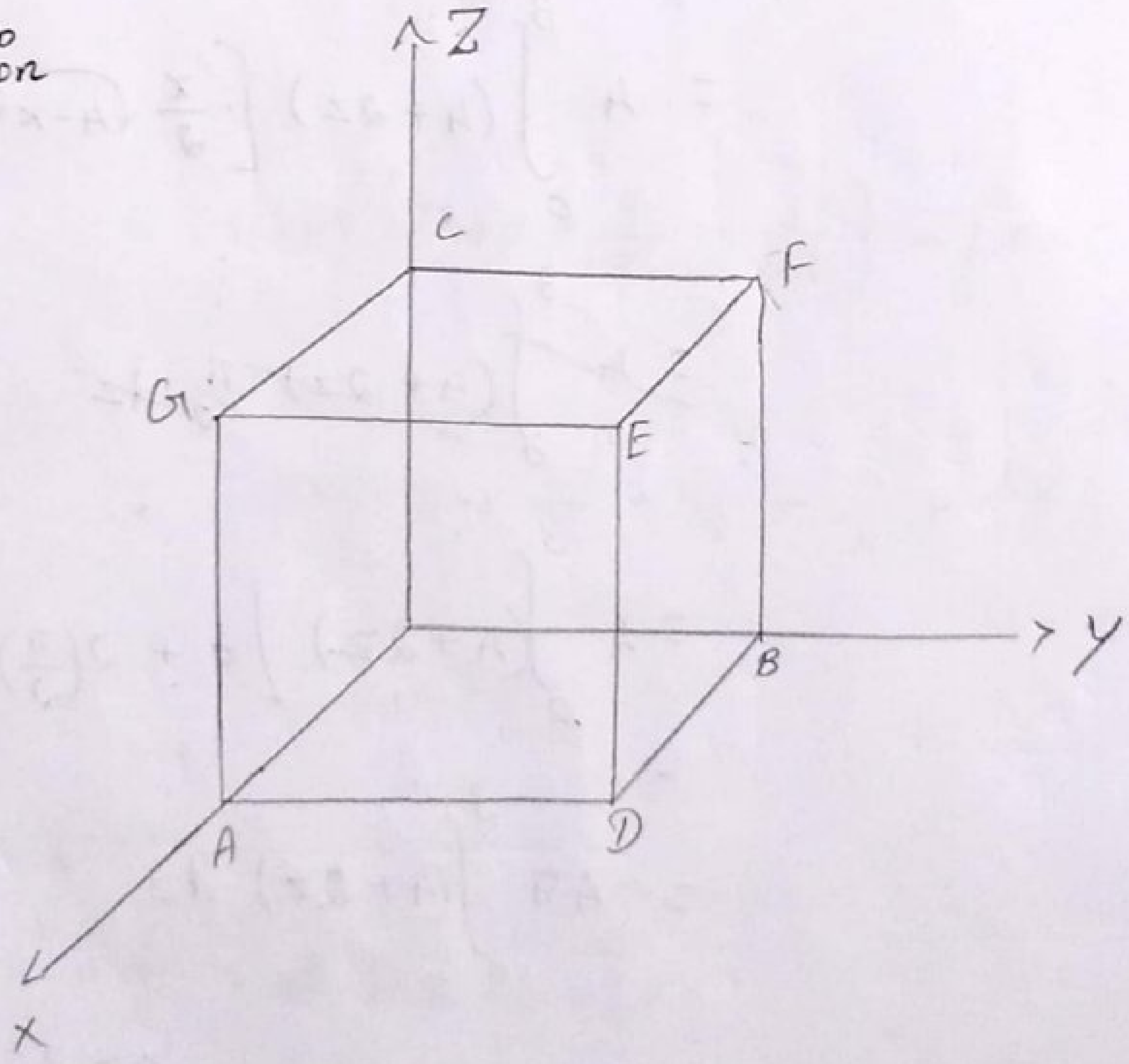
8. Verify the divergence theorem for the vector ~~$\vec{A} = (x^2 - y^2)\vec{i}$~~ $\vec{A} = (x^2 - yz)\vec{i} + (y^2 - zx)\vec{j} + (z^2 - xy)\vec{k}$ taken over the paralleopiped defined by

$0 \le x \le a, \; 0 \le y \le b, \; 0 \le z \le c.$

Solution



To Prove: $\iint_S \vec{A} \cdot \vec{ds} = \iiint_V \nabla \cdot \vec{A}\, dv$

$$\nabla \cdot \vec{A} = \frac{\partial}{\partial x}(x^2 - yz) + \frac{\partial}{\partial y}(y^2 - zx) + \frac{\partial}{\partial z}(z^2 - xy)$$

$$= 2x + 2y + 2z$$

$$= 2(x+y+z)$$

$$\iiint_V \nabla \cdot \vec{A}\, dv = 2\int_0^c\int_0^b\int_0^a (x+y+z)\, dx\, dy\, dz$$

$$= 2\int_0^c\int_0^b \left[\frac{x^2}{2} + yx + zx\right]_0^a dy\, dz$$

$$= 2\int_0^c\int_0^b \left[\frac{a^2}{2} + ay + az\right] dy\, dz$$

$$= 2\int_0^c \left[\frac{a^2 y}{2} + \frac{ay^2}{2} + azy\right]_0^b dz$$

$$= 2\int_0^c \left[\frac{a^2 b}{2} + \frac{ab^2}{2} + abz\right] dz$$

$$= 2\left[\frac{a^2 bz}{2} + \frac{ab^2 z}{2} + \frac{abz^2}{2}\right]_0^c$$

$$= 2\left[\frac{a^2 bc}{2} + \frac{ab^2 c}{2} + \frac{abc^2}{2}\right]$$

$$= a^2 bc + ab^2 c + abc^2$$

$$= abc\left[a + b + c\right] \longrightarrow ①$$

$$\iint_S \vec{A}\cdot\hat{n}\,ds = \iint_{OADB} + \iint_{EFCG} + \iint_{OBFG} + \iint_{ADEG}$$

$$+ \iint_{OACG} + \iint_{DBFE} (\vec{A}\cdot\hat{n}\,ds)$$

$$= \iint_{\substack{z=0\\ \hat{n}=-\vec{k}}} + \iint_{\substack{z=c\\ \hat{n}=\vec{k}}} + \iint_{\substack{x=0\\ \hat{n}=-\vec{i}}} + \iint_{\substack{x=a\\ \hat{n}=\vec{i}}} + \iint_{\substack{y=0\\ \hat{n}=-\vec{j}}} + \iint_{\substack{y=b\\ \hat{n}=\vec{j}}}$$

$$(\vec{A}\cdot\hat{n}\,ds)$$

$$= \int_0^b\int_0^a xy\,dx\,dy + \int_0^b\int_0^a (c^2-xy)\,dx\,dy + \int_0^c\int_0^b yz\,dy\,dz$$

$$+ \int_0^c\int_0^b (a^2-yz)\,dy\,dz + \int_0^c\int_0^a zx\,dx\,dz + \int_0^c\int_0^a (b^2-xz)\,dx\,dz$$

$$= \left[\frac{x^2}{2}\right]_0^a \left[\frac{y^2}{2}\right]_0^b + \int_0^b \left[c^2x - \frac{x^2}{2}y\right]_0^a dy + \left[\frac{y^2}{2}\right]_0^b$$

$$\left[\frac{z^2}{2}\right]_0^c + \int_0^c \left[a^2y - \frac{y^2}{2}z\right]_0^b dz + \left[\frac{z^2}{2}\right]_0^c \left[\frac{x^2}{2}\right]_0^a$$

$$+ \int_0^c \left[b^2x - \frac{x^2}{2}z\right]_0^a dz$$

$$= \frac{a^2b^2}{4} + \int_0^b \left[c^2a - \frac{a^2}{2}y\right] dy + \frac{b^2c^2}{4} +$$

$$\int_0^c \left[a^2b - \frac{b^2}{2}z\right] dz + \cancel{\frac{z^2a^2}{4}}\ \frac{c^2a^2}{4} + \int_0^c \left[b^2a - \frac{a^2}{2}z\right] dz$$

$$= \frac{a^2b^2}{4} + \left[c^2ay - \frac{a^2}{2}\frac{y^2}{2}\right]_0^b + \frac{b^2c^2}{4} +$$

$$\left[a^2bz - \frac{b^2}{2}\frac{z^2}{2}\right]_0^c + \frac{c^2a^2}{4} + \left[b^2az - \frac{a^2}{2}\frac{z^2}{2}\right]_0^c$$

$$= \cancel{\frac{a^2b^2}{4}} + abc^2 - \cancel{\frac{a^2b^2}{4}} + \cancel{\frac{b^2c^2}{4}} + a^2bc -$$

$$\cancel{\frac{b^2c^2}{4}} + \cancel{\frac{c^2a^2}{2}} + ab^2c - \cancel{\frac{a^2c^2}{4}}$$

$$= abc^2 + a^2bc + ab^2c$$

$$= abc(a+b+c) \longrightarrow ②$$

From ① and ②, we get

$$\iint_S \vec{A} \cdot \vec{ds} = \iiint_V \nabla \cdot \vec{A}\, ds$$

Hence GDT is verified.

---

9. Evaluate $\iint_S \vec{F} \cdot ds$ if $\vec{F} = x^3\vec{i} + y^3\vec{j} + z^3\vec{k}$ and S is the surface of the sphere $x^2+y^2+z^2=a^2$

Solution:

By GDT, we have

$$\iint_S \vec{F} \cdot \vec{ds} = \iiint_V \nabla \cdot \vec{F}\, dv$$

$$\nabla \cdot \vec{F} = \frac{\partial}{\partial x}(x^3) + \frac{\partial}{\partial y}(y^3) + \frac{\partial}{\partial z}(z^3)$$

$$= 3x^2+3y^2+3z^2$$

$$= 3(x^2+y^2+z^2)$$

$= 3a^2$

Put $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$

$dx\,dy\,dz = r^2\sin\theta\,dr\,d\theta\,d\phi.$

$$\therefore \iint_S \vec{F}\cdot\vec{ds} = \int_{r=0}^{a}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} 3a^2 r^2\sin\theta\,dr\,d\theta\,d\phi$$

$$= 3a^2\int_0^a\int_0^\pi r^2\sin\theta\,dr\,d\theta[\phi]_0^{2\pi}$$

$$= 3a^2\int_0^a\int_0^\pi r^2\sin\theta\,dr\,d\theta\,(2\pi)$$

$$= 6a^2\pi\int_0^a\int_0^\pi r^2\sin\theta\,dr\,d\theta$$

$$= 6a^2\pi\int_0^a r^2[-\cos\theta]_0^\pi\,dr$$

$$= 6a^2\pi\int_0^a r^2(1+1)\,dr$$

$$= 12a^2\pi\int_0^a r^2\,dr$$

$$= 12a^2\pi\left[\frac{r^3}{3}\right]_0^a$$

$$= 4a^2\pi(a^3)$$

$$= 4\pi a^5$$

10. Evaluate by Green's Theorem

$\int_C (xy+z^2)\,dx + (x^2+y^2)\,dy$ where C is the

sequare bounded by the lines $x=\pm 1$, $y=\pm 1$,

in the xoy plane.

Solution:

We have by Green's Theorem

$$\int_C (M\,dx + N\,dy) = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

Now

$$M = xy + z^2 \qquad , \qquad N = x^2+y^2$$

$$\frac{\partial M}{\partial y} = x \qquad \frac{\partial N}{\partial x} = 2x$$

$$\therefore \int_C M\,dx + N\,dy = \int_{-1}^{1}\int_{-1}^{1} (2x-x)\,dx\,dy$$

$$= \int_{-1}^{1}\int_{-1}^{1} x\,dx\,dy$$

$$= \left[\frac{x^2}{2}\right]_{-1}^{1} \left[y\right]_{-1}^{1}$$

$$= \left[\frac{1}{2} + \frac{1}{2}\right]\left[1-1\right]$$

$$= 0$$

---

11) Using Green's Theorem, evaluate

$\int_C (3x^2 - 8y^2)\,dx + (4y - 6xy)\,dy$ where C is the

boundery of the rectangular area enclosed by the lines $x=0$, $x=1$ and $y=0$, $y=2$.

Solution:

We have by Green's Theorem

$$\int_C M dx + N dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\, dy$$

Now,

$$M = 3x^2 - 8y^2 \qquad N = 4y - 6xy$$

$$\frac{\partial M}{\partial y} = -16y \qquad \frac{\partial N}{\partial x} = -6y$$

$$\therefore \int_C M dx + N dy = \int_0^1\int_0^2 (-6y+16y)\, dx\, dy$$

$$= \int_0^1\int_0^2 10y\, dx\, dy$$

$$= [x]_0^1 \left[\frac{10y^2}{2}\right]_0^2$$

$$= (1)\,(5y^2)_0^2$$

$$= 20$$

---

12. Prove that the area enclosed by a simple closed curve C in xoy plane is $\frac{1}{2}\int_C x dy - y dx$. deduce the area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Solution:

By Green's Theorem, we have

$$\int_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

Take $M = -y \quad N = x$

$$\frac{\partial M}{\partial y} = -1 \quad , \quad \frac{\partial N}{\partial x} = 1$$

$$\int_C M\,dx + N\,dy = \iint_R (1+1)\,dx\,dy$$

(i.e) $\int_C x\,dy - y\,dx = 2\iint_R dx\,dy$

$$= 2\,(\text{area of } R)$$

$\therefore$ Area n R $= \frac{1}{2}\int_C x\,dy - y\,dx$

Let $x = a\cos\theta \quad , y = b\sin\theta$

Then $dx = -a\sin\theta\,d\theta \quad , \quad dy = b\cos\theta\,d\theta$

Area of ellipse $= \frac{1}{2}\int_C x\,dy - y\,dx$

$$= \frac{1}{2}\int_0^{2\pi} a\cos\theta\,(b\cos\theta\,d\theta) - b\sin\theta\,(-a\sin\theta\,d\theta)$$

$$= \frac{1}{2}ab\int_0^{2\pi} \cos^2 d\theta + \sin^2\theta\,d\theta$$

$$= \frac{1}{2}ab\int_0^{2\pi} d\theta$$

$$= \frac{1}{2}ab\,[\theta]_0^{2\pi}$$

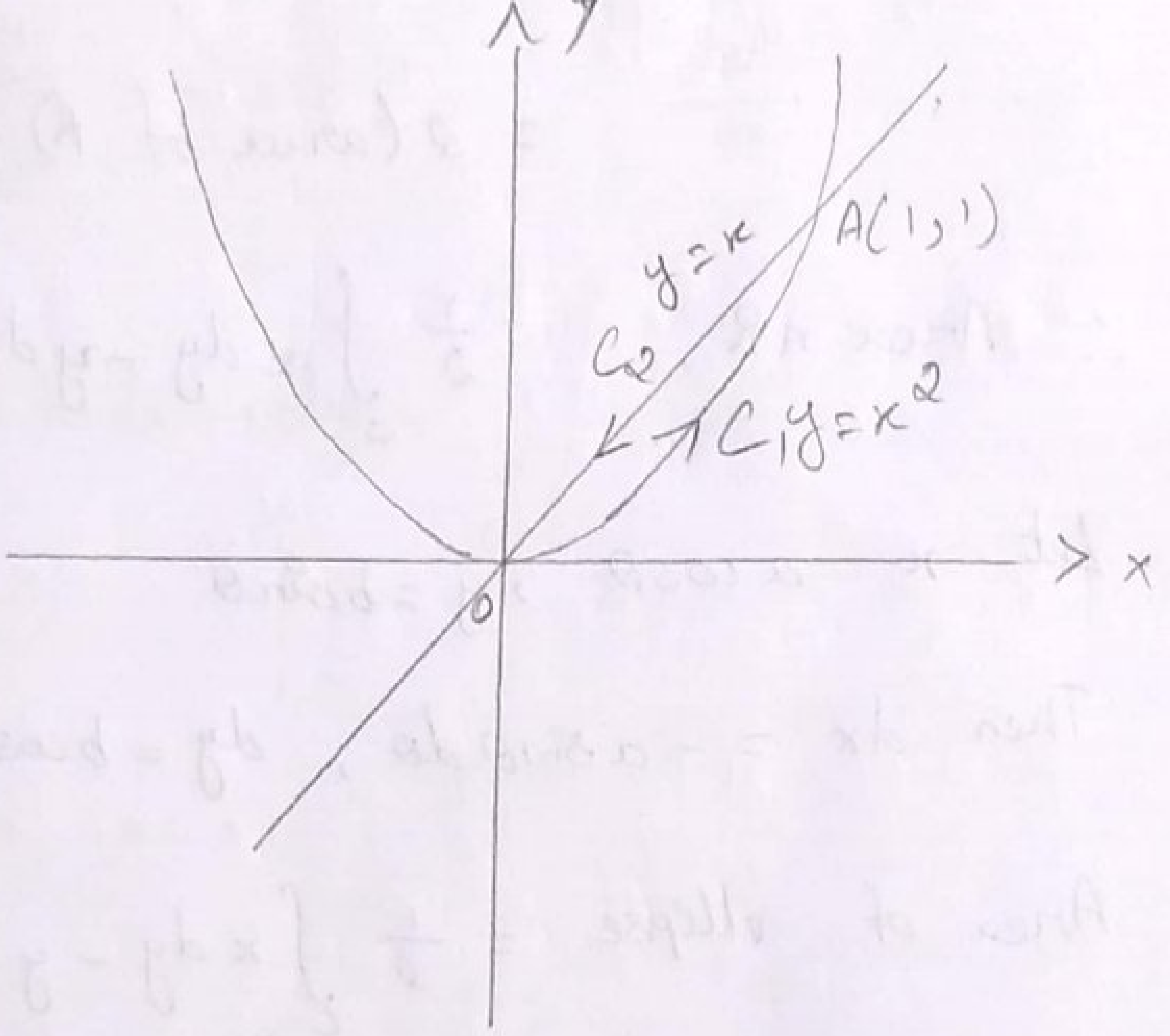
$$= \frac{1}{2}ab\,(2\pi)$$

$$= ab\pi$$

13. Verify Green's Theorem in plane for the integral $\int (xy+y^2)\,dx + x^2\,dy$, where C is the enclosed curve enclosing the region R bounded by the parabola $y=x^2$ and the line $y=x$.

Solution:

To verify $\int_C M dx + N dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx\, dy$



$$y = x \longrightarrow ①$$

$$y = x^2 \longrightarrow ②$$

Solve ① and ②

$$x = x^2$$

$$x^2 - x = 0$$

$$x(x-1) = 0 \Rightarrow x = 0, \quad x = 1$$

$$x = 0 \Rightarrow y = 0$$

$$x = 1 \Rightarrow y = 1$$

Point of intersections are

$O(0,0)$, $A(1,1)$

$$\int_C (xy+y^2)\,dx + x^2\,dy = \int_{C_1} + \int_{C_2} \left\{(xy+y^2)dx + x^2 dy\right\}$$

$$= \int_0^1 \left[(x x^2 + x^4)\,dx + x^2 d\, 2x dx\right] +$$

$$\int_1^0 \left[(x^2+x^2)\,dx + x^2 dx\right]$$

$$= \int_0^1 \left[x^3 + x^4 + 2x^3\right] dx + \cancel{\int_0^1 2x^2 + x} \int_1^0 (2x^2+x^2)\,dx$$

$$= \int_0^1 \left[3x^3 + x^4\right] dx + \int_1^0 3x^2\,dx$$

$$= \left[3\frac{x^4}{4} + \frac{x^5}{5}\right]_0^1 + \left[\frac{3x^3}{3}\right]_1^0$$

$$= \frac{3}{4} + \frac{1}{5} - 0 + 0 - 1$$

$$= \frac{3}{4} + \frac{1}{5} - 1$$

$$= \frac{15+4-20}{20}$$

$$= \frac{-1}{20}$$

(ie) $\int_C M\,dx + N\,dy = \frac{-1}{20}$ ⟶ ③

$M = xy + y^2$, $N = x^2$

$$\frac{\partial M}{\partial y} = x + 2y \qquad \frac{\partial N}{\partial x} = 2x$$

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx\,dy = \int_0^1 \int_{x^2}^{x} (2x - x - 2y)\, dx\,dy$$

$$= \int_0^1 \int_{x^2}^{x} (x - 2y)\, dx\,dy$$

$$= \int_0^1 \left[xy - \frac{2y^2}{2} \right]_{x^2}^{x} dx$$

$$= \int_0^1 \left[\left(x\,x - \frac{2x^2}{2} \right) - \left(x\,x^2 - \frac{2x^4}{2} \right) \right] dx$$

$$= \int_0^1 \left[(x^2 - x^2) - (x^3 - x^4) \right] dx$$

$$= \int_0^1 (x^4 - x^3)\, dx$$

$$= \left[\frac{x^5}{5} - \frac{x^4}{4} \right]_0^1 = \frac{1}{5} - \frac{1}{4}$$

$$= \frac{4-5}{20}$$

$$= \frac{-1}{20} \longrightarrow ④$$

From ③ and ④, we get

$$\int_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial y} - \frac{\partial M}{\partial x} \right) dx\,dy$$

Hence Green's theorem is verified.

---

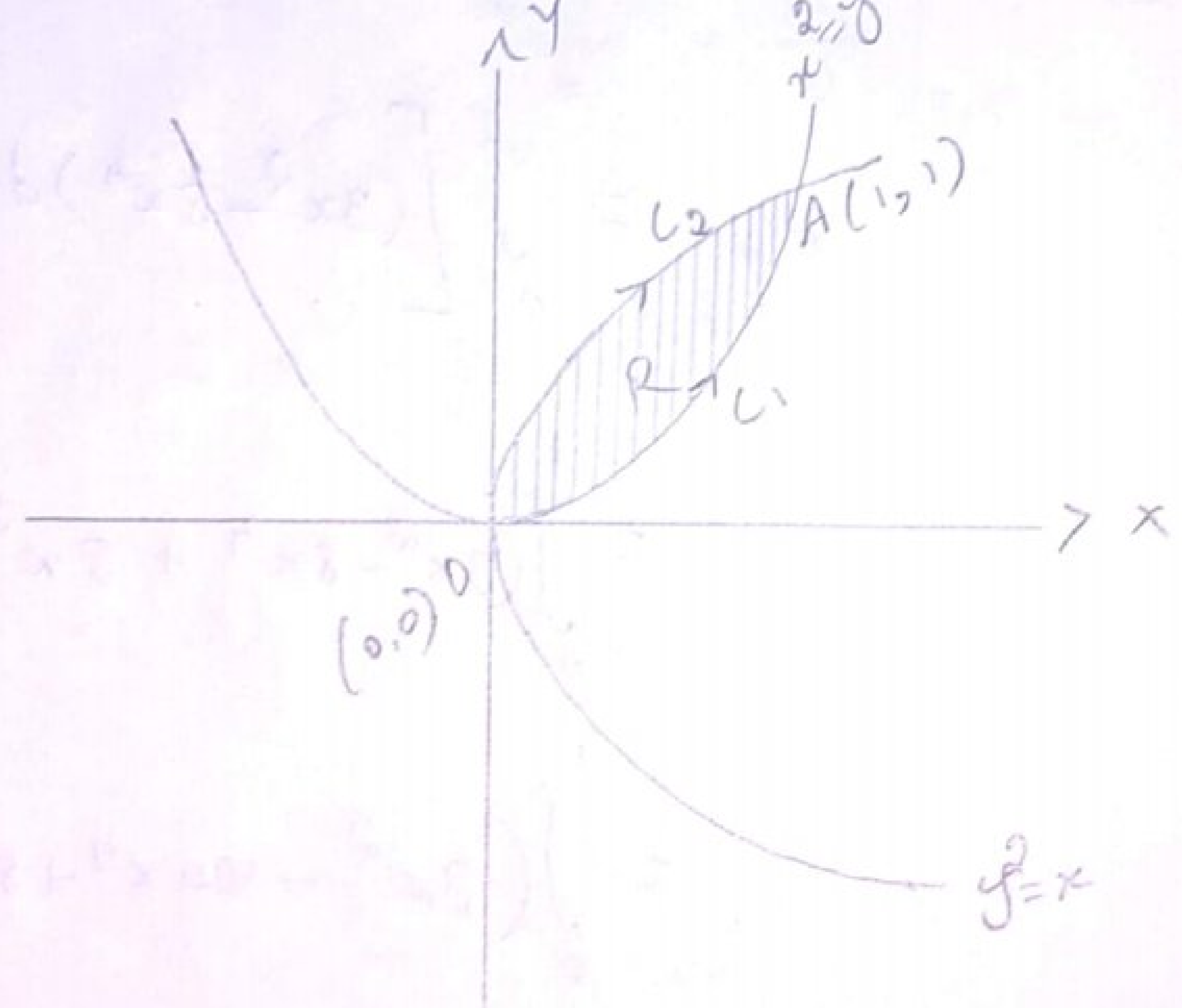
4. Verify Green's Theorem for

$\int_C (3x^2 - 8y^2)\,dx + (4y - 6xy)\,dy$ where C is the boundary of the ~~so~~ region R enclosed by the parabolas $y = x^2$ and $y^2 = x$.

Solution:

Green's Theorem is

$$\int_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx\,dy$$



$$y = x^2 \longrightarrow ①$$

$$y^2 = x \longrightarrow ②$$

Solve ① and ②

~~$x^4 = x$~~ $x^4 = x$

$x^4 - x = 0$

$x(x^3 - 1) = 0 \Rightarrow x = 0, \quad x = 1$

$x=0 \Rightarrow y=0$

$x=1 \Rightarrow y=1$

$\therefore$ Point of intersections are $(0,0)$ $(1,1)$

$M = 3x^2 - 8y^2$, $N = 4y - 6xy$

$$\int_C (M\,dx + N\,dy) = \int_{C_1} + \int_{C_2} (M\,dx + N\,dy)$$

Along C_1 $x^2 = y$

$$\int_{C_1} (M\,dx + N\,dy) = \int_0^1 \left[(3x^2-8y^2)dx + (4y-6xy)dy\right]$$

$$= \int_0^1 \left[(3x^2-8x^4)dx + (4x^2-6x^3)\underset{2x\,dx}{\cancel{dy}}\right]$$

$$= \int_0^1 (3x^2-8x^4+8x^3-12x^4)\,dx$$

$$= \int_0^1 (3x^2-20x^4+8x^3)\,dx$$

$$= \left[\frac{3x^3}{3} - \frac{20x^5}{5} + \frac{8x^4}{4}\right]_0^1$$

$$= \left[x^3 - 4x^5 + 2x^4\right]_0^1$$

$$= 1-4+2$$

$$= -1$$

Along C_2 , $y^2 = x$

$$\int_{C_2} (M\,dx + N\,dy) = \int_1^0 \left[(3x^2 - 8y^2)\,dx + (4y - 6xy)\,dy\right]$$

$$= \int_1^0 \left[(3y^4 - 8y^2)\,2y\,dy + (4y - 6y^3)\,dy\right]$$

$$= \int_1^0 \left[6y^5 - 16y^3 + 4y - 6y^3\right] dy$$

$$= \int_1^0 (6y^5 - 22y^3 + 4y)\,dy$$

$$= \left[\frac{6y^6}{6} - \frac{22y^4}{4} + \frac{4y^2}{2}\right]_1^0$$

$$= \left[y^6 - \frac{11y^4}{2} + 2y^2\right]_1^0$$

$$= (0) - \left[1 - \frac{11}{2} + 2\right]$$

$$= -\left[\frac{2 - 11 + 4}{2}\right]$$

$$= -\left[\frac{6 - 11}{2}\right] = -\left(\frac{-5}{2}\right)$$

$$= \frac{5}{2}$$

$$\therefore \int_C (M\,dx + N\,dy) = -1 + \frac{5}{2} = \frac{3}{2} \longrightarrow ③$$

$$\frac{\partial M}{\partial y} = -16y \qquad \frac{\partial N}{\partial x} = -6y$$

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \int_0^1 \int_{x^2}^{\sqrt{x}} (-6y + 16y)\, dx\,dy$$

$$= \int_0^1 \int_{x^2}^{\sqrt{x}} 10y \, dx\,dy$$

$$= 10 \int_0^1 \left[\frac{y^2}{2}\right]_{x^2}^{\sqrt{x}} dx = 10\int_0^1 \left[\frac{x}{2} - \frac{x^4}{2}\right] dx$$

$$= 5 \int_0^1 (x - x^4)\, dx = 5\left[\frac{x^2}{2} - \frac{x^5}{5}\right]_0^1$$

$$= 5\left[\frac{1}{2} - \frac{1}{5}\right] = 5\left[\frac{5-2}{10}\right]$$

$$= 5\left(\frac{3}{10}\right) = \frac{3}{2} \longrightarrow ④$$

From ③ and ④ we get

$$\cancel{\int_C (M\,dx + N\,dy)} = \iint_R \cancel{\frac{dM}{dx}} \int$$

$$\int_C (M\,dx + N\,dy) = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy.$$

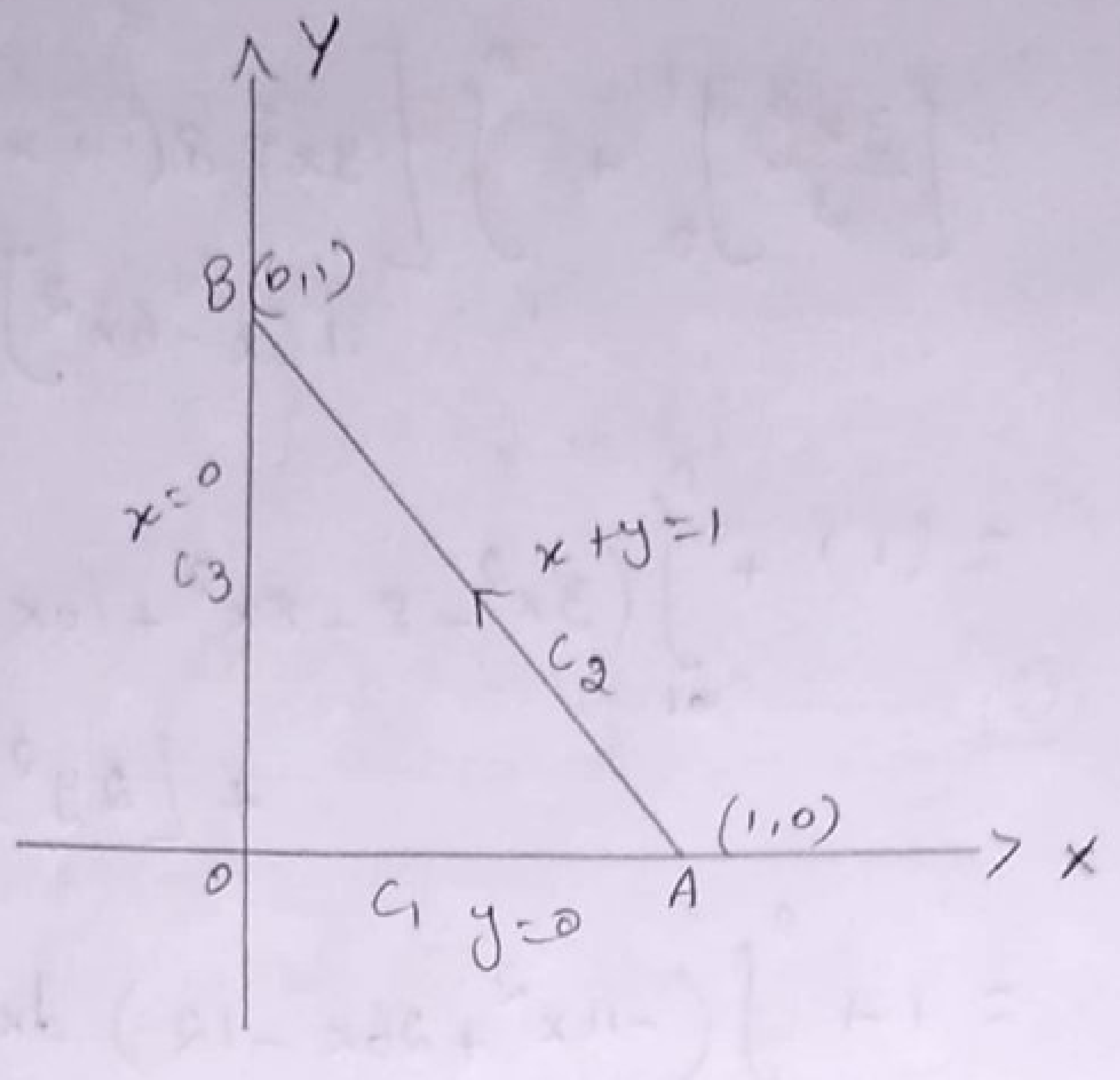
15. Verify Green's Theorem for

$\iint_C \left[(3x^2 - 8y^2)dx + (4y - 6xy)\,dy\right]$ where C is the

boundary of the region R enclosed by

$x = 0$, $y = 0$, $x + y = 1$

Solution:



$$M = 3x^2 - 8y^2 \qquad N = 4y - 6xy$$

$$M\,dx + N\,dy = (3x^2 - 8y^2)\,dx + (4y - 6xy)\,dy$$

$$\frac{\partial M}{\partial y} = -16y \qquad \frac{\partial N}{\partial x} = -6y$$

$y = 1 - x$

$dy = -dx$

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = -6y + 16y = 10y$$

~~$\int_C (M\,dx + N\,dy) = \int_{C_1} + \int_{C_2} + \int_{C_3} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$~~

$$\int_C (M\,dx + N\,dy) = \int_{C_1} + \int_{C_2} + \int_{C_3} (M\,dx + N\,dy)$$

$$= \int_0^1 3x^2\,dx + \int_1^0 \Big[3x^2 - 8(1-x)^2\,dx + 4(1-x) - 6x(1-x)(-dx)\Big] + \int_1^0 4y\,dy$$

~~$= \left[\frac{3x^3}{3}\right]_0^1 + \int_1^0 \left[\left(3x^2 - 8(1+x^2-2x)\right) dx\right.$~~

$$= \left[\frac{3x^3}{3}\right]_0^1 + \int_1^0 \left[3x^2 - 8(1+x^2-2x) - 4 + 4x + 6x - 6x^2\right] dx + \left[4\frac{y^2}{2}\right]_1^0$$

$$= (1) + \int_1^0 (3x^2 - 8 - 8x^2 + 16x - 4 + 10x - 6x^2) + \left[2y^2\right]_1^0$$

$$= 1 + \int_1^0 (-11x^2 + 26x - 12)\, dx + (0 - 2)$$

$$= 1 + \left[\frac{-11x^3}{3} + \frac{26x^2}{2} - 12x\right]_1^0 - 2$$

$$= \left[\frac{-11}{3}x^3 + 13x^2 - 12x\right]_1^0 - 1$$

$$= \left[0 - \left(\frac{-11}{3} + 13 - 12\right)\right] - 1$$

$$= \frac{11}{3} - 13 + 12 - 1 = \frac{11 - 39 + 36 - 3}{3}$$

$$= \frac{47 - 42}{3} = \frac{5}{3} \longrightarrow ①$$

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\, dy = \int_0^1 \int_0^{1-x} 10y\, dx\, dy$$

$$= 10 \int_0^1 \left[\frac{y^2}{2}\right]_0^{1-x} dx$$

$$= \frac{10}{2} \int_0^1 (1-x)^2 \, dx = 5 \int_0^1 (1+x^2-2x) \, dx$$

$$= 5 \left[x + \frac{x^3}{3} - \frac{2x^2}{2} \right]_0^1$$

$$= 5 \left[1 + \frac{1}{3} - 1 \right]$$

$$= 5 \left(\frac{1}{3} \right) = \frac{5}{3} \longrightarrow ②$$

From ① and ②, we get

$$\int_C (M\,dx + N\,dy) = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx\,dy$$

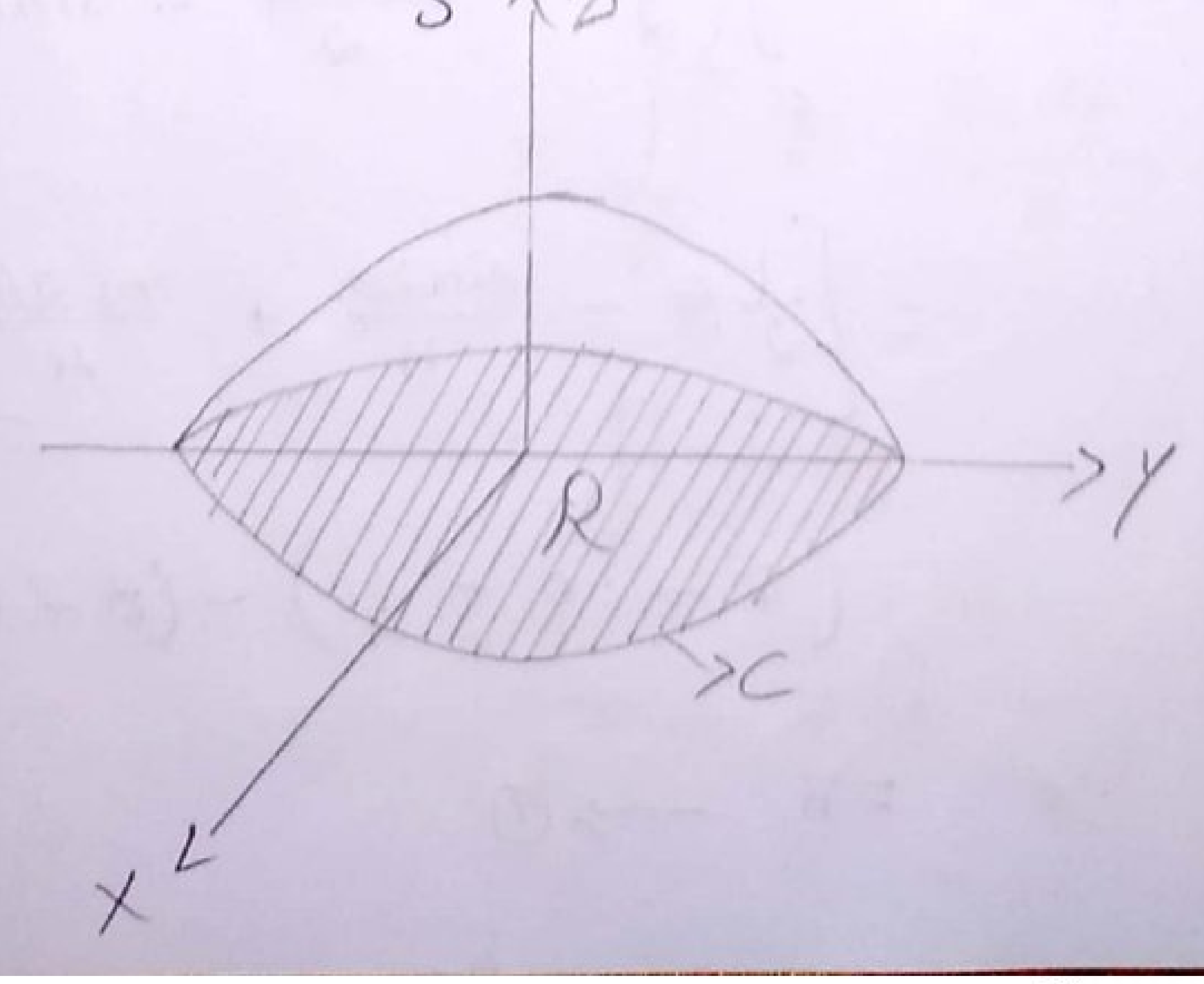
Hence Green's theorem is verified.

---

16. Verify stoke's theorem for $\vec{A} = (2x-y)\vec{i} - yz^2\vec{j} - y^2z\vec{k}$ taken over the upper half ~~of~~ surface of the sphere $x^2+y^2+z^2=1$, $z \geq 0$ and the boundary curve C circle $x^2+y^2 \pm 1$, $z=0$.

Solution:

$$\int_C \vec{A} \cdot \vec{dr} = \iint_S (\nabla \times \vec{A}) \hat{n} \, ds$$



Equation of circle is

$$x^2+y^2=1 \quad , z=0$$

$$x=\cos\theta \,, \quad y=\sin\theta \,, \quad z=0$$

$$dx=-\sin\theta\, d\theta \,, dy=\cos\theta\, d\theta \,, \quad dz=0$$

~~θ various~~ -

θ varies from 0 to 2π

$$\vec{A}\cdot\vec{dr} = (2x-y)\,dx - yz^2dy - y^2z\,dz$$

$$=(2\cos\theta-\sin\theta)(-\sin\theta\, d\theta) - 0 - 0$$

$$= \cancel{-2\sin\theta}$$

$$=(-2\sin\theta\cos\theta + \sin^2\theta)\, d\theta$$

$$=(\sin^2\theta - 2\sin\theta\cos\theta)\, d\theta$$

$$=\left(\frac{1-\cos 2\theta}{2} - \sin 2\theta\right) d\theta$$

$$=\left(\frac{1}{2} - \frac{\cos 2\theta}{2} - \sin 2\theta\right) d\theta$$

$$\int_C \vec{A}\cdot\vec{dr} = \int_0^{2\pi}\left(\frac{1}{2} - \frac{\cos 2\theta}{2} - \sin 2\theta\right) d\theta$$

$$=\left[\frac{1}{2}\theta - \frac{\sin 2\theta}{4} + \frac{\cos 2\theta}{4}\right]_0^{2\pi}$$

$$=\left(\pi - 0 + \frac{1}{4}\right) - \left(0 - 0 + \frac{1}{4}\right)$$

$$=\pi \longrightarrow ①$$

Let R be the projection of S on xoy plane.

$$\phi = x^2+y^2+z^2-1$$

$$\nabla\phi = 2x\vec{i} + 2y\vec{j} + 2z\vec{k}$$

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2x\vec{i}+2y\vec{j}+2z\vec{k}}{\sqrt{4x^2+4y^2+4z^2}}$$

$$= \frac{2x\vec{i}+2y\vec{j}+2z\vec{k}}{2(1)} = x\vec{i}+y\vec{j}+z\vec{k}$$

$$\nabla\times\vec{A} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x-y & -yz^2 & -y^2z \end{vmatrix}$$

$$= \vec{i}\left[-2yz+2yz\right] - \vec{j}\left[0-0\right] + \vec{k}\left[0+1\right]$$

$$= \vec{i}(0) - \vec{j}(0) + \vec{k}$$

$$= \vec{k}$$

$$\iint_S (\nabla\times\vec{A})\cdot\hat{n}\,ds = \iint_R z\,\frac{dx\,dy}{|\hat{n}\cdot\vec{k}|}$$

$$= \iint_R z\,\frac{dx\,dy}{z}$$

$$= \iint_R dx\,dy$$

= area of k

$= \pi(1)^2$

$= \pi \longrightarrow ②$

$\therefore$ ① and ②

$$\int_C \vec{A}\cdot\vec{dr} = \iint_S (\nabla\times\vec{A})\cdot\hat{n}\,ds$$

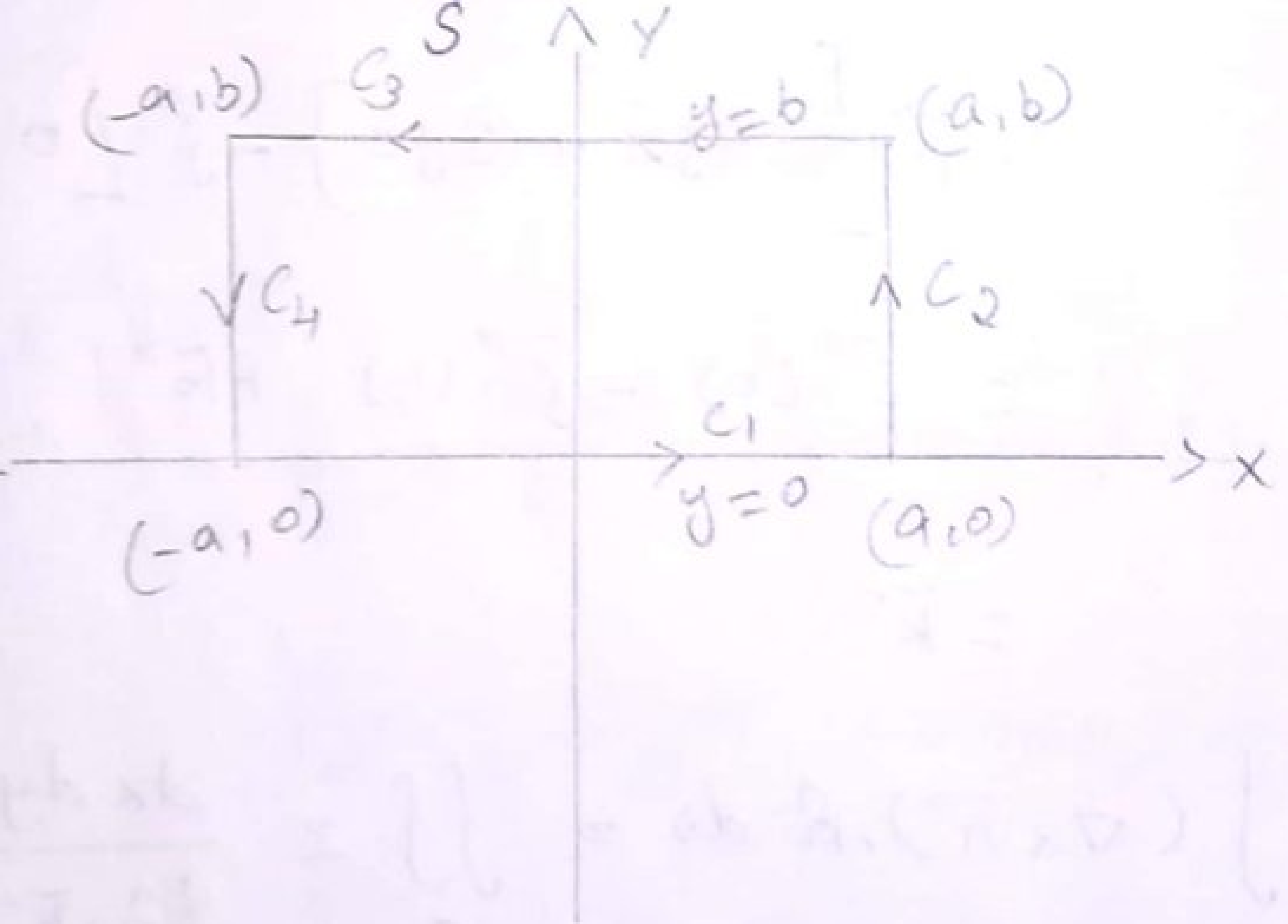
Hence stoke's theorem is verified.

---

17. Verify stoke's theorem for $\vec{F} = (x^2+y^2)\,\vec{i} - 2xy\,\vec{j}$ taken over the rectangle bounded by the lines $x = \pm a$, $y=0$, $y=b$.

Solution:

$$\int_C \vec{F}\cdot\vec{dr} = \iint_S (\nabla\times\vec{F})\cdot\hat{n}\,ds$$



$$\vec{F}\cdot\vec{dr} = (x^2+y^2)\,dx - 2xy\,dx$$

$$\int_C \vec{F}\cdot\vec{dr} = \int_{C_1} + \int_{C_2} + \int_{C_3} + \int_{C_4} (\vec{F}\cdot\vec{dr})$$

$$= \int_{y=0} + \int_{x=a} + \int_{y=b} + \int_{x=-a} (\vec{F}\cdot\vec{dr})$$

$$= \int_{-a}^{a} x^2\,dx + \int_{0}^{b} (-2ay)\,dy + \int_{a}^{-a} (x^2+b^2)\,dx + \int_{b}^{0} 2ay\,dy$$

$$= \left[\frac{x^3}{3}\right]_{-a}^{a} + \left[-2a\frac{y^2}{2}\right]_{0}^{b} + \left[\frac{x^3}{3} + b^2x\right]_{a}^{-a} + \left[\frac{2a\,y^2}{2}\right]_{b}^{0}$$

~~$= \left[\frac{a^3}{3} + \frac{a^3}{3}\right] - 2a$~~

$$= \frac{a^3}{3} + \frac{a^3}{3} - ab^2 + \left[\frac{-a^3}{3} + b^2(-a) - \frac{a^3}{3} - ab^2\right] - ab^2$$

$$= \frac{\cancel{2a^3}}{\cancel{3}} - ab^2 - \frac{\cancel{a^3}}{\cancel{3}} - ab^2 - \frac{\cancel{a^3}}{\cancel{3}} - ab^2 - ab^2$$

$$= -4ab^2 \longrightarrow ①$$

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2+y^2 & -2xy & 0 \end{vmatrix}$$

$$= \vec{i}\,[0-0] - \vec{j}\,[0-0] + \vec{k}\,[-2y-2y]$$

$$= -4y\,\vec{k}$$

$$\hat{n} = \vec{k}$$

$$\iint_S (\nabla \times \vec{F}) \cdot \hat{n} \, ds = \iint_R -4y \frac{dx\,dy}{1}$$

$$= -4 \int_{-a}^{a} \int_{0}^{b} y \, dx \, dy$$

$$= -4 \left[x\right]_{-a}^{a} \left[\frac{y^2}{2}\right]_0^b$$

$$= -4(a+a)\left(\frac{b^2}{2}\right)$$

$$= -2(2a)(b^2)$$

$$= -4ab^2 \longrightarrow ②$$

From ① and ②, we get,

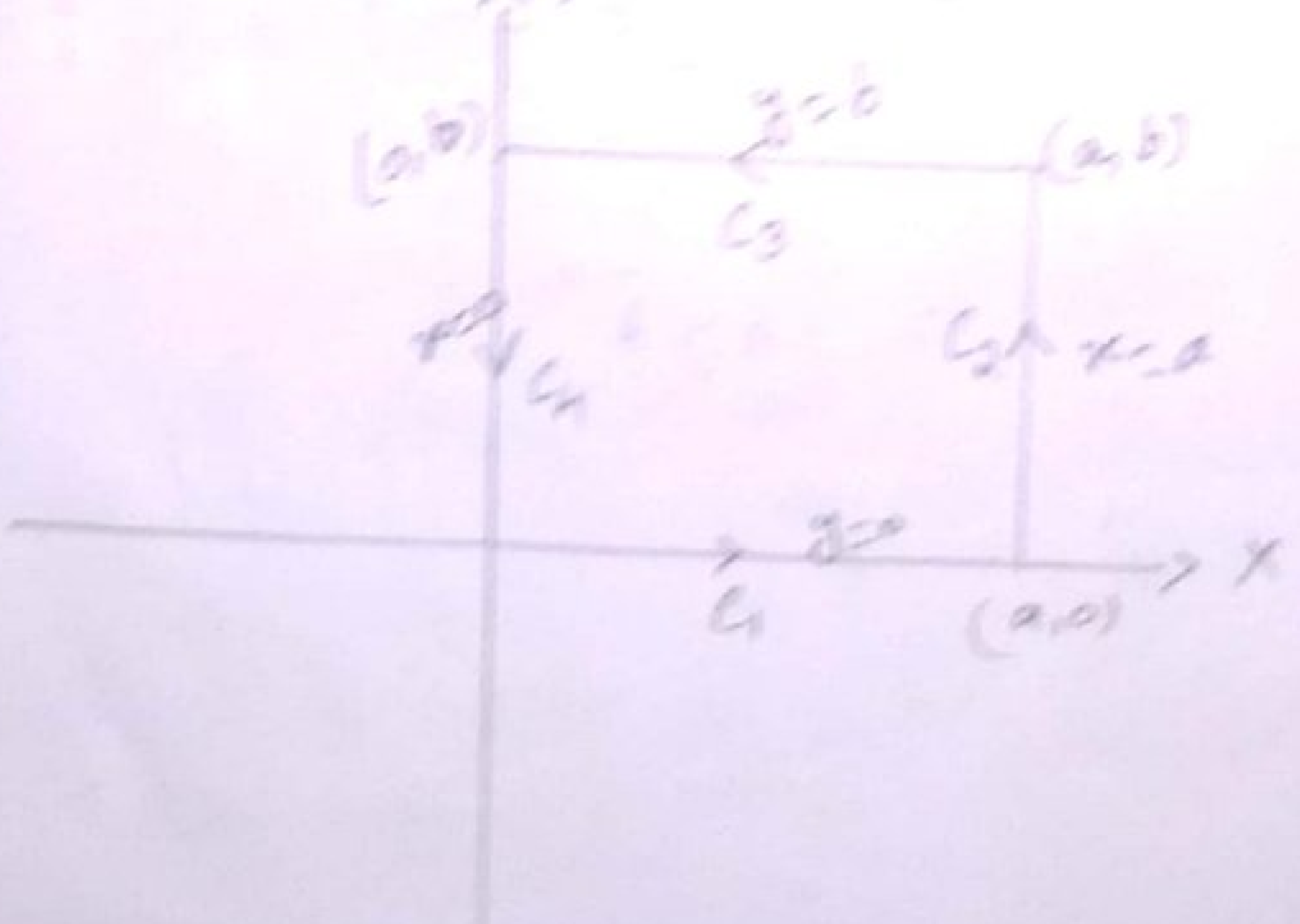
$$\int_C \vec{F} \cdot \vec{dr} = \iint_S (\nabla \times \vec{F}) \cdot \hat{n} \, ds$$

---

18. Verify stoke's theorem for $\vec{F} = (x^2 - y^2)\vec{i} + 2xy\vec{j}$ taken over the rectangle bounded by the lines $x=0$, $x=a$; $y=0$, $y=b$.

Solution:

$$\int_C \vec{F} \cdot \vec{dr} = \iint_S (\nabla \times \vec{F}) \cdot \hat{n} \, ds$$



$$\vec{F}\cdot\vec{dr} = (x^2-y^2)\,dx + 2xy\,dy$$

$$\int_C \vec{F}\cdot\vec{dr} = \int_{C_1} + \int_{C_2} + \int_{C_3} + \int_{C_4} (\vec{F}\cdot\vec{dr})$$

$$= \int_{y=0} + \int_{x=a} + \int_{y=b} + \int_{x=0} (\vec{F}\cdot\vec{dr})$$

$$= \int_0^a x^2\,dx + \int_0^b 2ay\,dy + \int_a^0 (x^2-b^2)\,dx + \int_b^0 0\,dy$$

$$= \left[\frac{x^3}{3}\right]_0^a + \left[\frac{2ay^2}{2}\right]_0^b + \left[\frac{x^3}{3} - b^2x\right]_a^0 + 0$$

$$= \frac{a^3}{3} + ab^2 + b^2a - \frac{a^3}{3}$$

$$= 2ab^2 \longrightarrow ①$$

$$\nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \frac{\partial}{\partial z} \\ x^2-y^2 & 2xy & 0 \end{vmatrix}$$

$$= \vec{i}\,[0-0] - \vec{j}\,[0-0] + \vec{k}\,[2y+2y]$$

$$= 4y\,\vec{k}$$

$$\hat{n} = \vec{k}$$

$$\iint_S (\nabla\times\vec{F})\cdot\hat{n}\,ds = \int_0^a\int_0^b 4y\,dx\,dy$$

$$= 4\,[x]_0^a\left[\frac{y^2}{2}\right]_0^b$$

$$= 4a\left(\frac{b^2}{2}\right)$$

$$= 2ab^2 \longrightarrow ②$$

From ① and ②, we get

$$\int_C \vec{F}\cdot d\vec{r} = \iint_S (\nabla\times\vec{F})\cdot\hat{n}\; ds$$

Theorem:

$$\iint (\phi\,\nabla\psi)\cdot ds = \iiint_V \left(\phi\,\nabla^2\psi + \nabla\phi\cdot\nabla\psi\right) dv$$

Proof:

[These results are called Green's first and second identities]

Setting $A = \phi(\nabla\psi)$

$$\iint_S (\phi\,\nabla\psi)\; ds = \iiint_V \nabla\cdot(\phi\,\nabla\psi)\; dv$$

$$\cancel{= \iiint_V (\nabla\phi)\,\nabla\psi)}$$

$$= \iiint_V \left[(\nabla\phi)(\nabla\psi) + \phi\,(\nabla\cdot\nabla\psi)\right] dv$$

$$= \iiint_V \left[(\nabla\phi)(\nabla\psi) + \phi\,(\nabla^2\psi)\right] dv \longrightarrow ①$$

Interchanging ϕ and ψ in ①, we get,

$$\iint_S \psi \nabla\phi \, ds = \iiint_V \left[(\nabla\psi).(\nabla\phi) + \psi\nabla^2\phi\right] dv \quad \longrightarrow ②$$

Now ① and ~~b~~ ② gives the results as stokes.

Part - A

1. If V is the volume of the region enclosed by the surface S of the sphere $x^2+y^2+z^2=a^2$, there $\iint \vec{r}.\vec{ds} =$ ______

a) V b) 2V [✓ c) 3V] d) 4V

2. If C is the circle $x^2+y^2=1$, then $\int (x-2y)\,dx + x\,dy$ is ______

a) π b) 2π c) 3π d) 4π

3. The jacobian of the transformation $x = r\cos\theta$ and $y = r\sin\theta$ is ______

a) r b) $r\cos\theta$ [✓ c) $r\sin\theta$] d) none of the above

4. $\iint_S \vec{A}.\vec{ds} = \iiint_V (\nabla.\vec{A})\,dv$ is ______

[✓ a) GDT] b) Green's Theorem c) Stoke's Theorem

d) none of the above.

5. The area bounded by a surface closed curve C in xy plane is ______

[✓ a)] $\frac{1}{2}\int_C x\,dy - y\,dx$ b) $\int_C x\,dy - y\,dx$

c) $\int_C x\,dy + y\,dx$ d) $\int_0 x\,dy + y\,dx$

6. The volume integral of A is enclosed by

a) $\int_C \vec{A}\cdot d\vec{r}$ b) $\iint_S \vec{A}\cdot d\vec{s}$ [✓ c)] $\iiint_V \vec{A}\, d\vec{v}$ d) $\iint_V \vec{A}\, d\vec{v}$

7. From the divergence theorem $\iint_S \phi\, ds$ = ______

a) $\int_C \nabla\phi\cdot d\vec{r}$ b) $\iiint_V \nabla\phi\cdot d\vec{v}$

c) $-\iiint_V \nabla\phi\cdot d\vec{v}$ d) $\iiint_V \nabla\phi\cdot d\vec{v}$

8) Green's Theorem .

8. Green's first identity is ______

[✓ a)] $\iint_S \phi(\nabla\psi)\, ds = \iiint_V (\phi\nabla^2\psi + \nabla\phi\cdot\nabla\psi)\, dv$

b) $\oint \vec{A}\cdot d\vec{r} = \iint_S (\nabla\times\vec{A})\, ds$

c) $\iint_S (\phi\nabla\psi)\, ds = \iiint_V (\phi\nabla^2\psi - \nabla\phi\cdot\nabla\psi)\, dv$

d) $\iint_S ((\phi\nabla\psi) - (\psi\nabla\phi))\, ds = \iiint_V (\phi\nabla^2\psi - \psi\nabla^2\phi)\, dv$

9. Prove stoke's Theorem , $\int_C \phi \, d\vec{r}$

a) $\iint_R \nabla\phi \, dr$ b) $\iint_S \nabla\phi \, d\vec{r}$

c) $-\iint_S (\nabla\phi) \times ds$ d) $-\iint_S (\nabla\phi) \times ds$

10. Find the value of divergence theorem for $\vec{A} = xy^2 \vec{i} + 6y^3 \vec{j} + y^2 z \vec{k}$ for a cuboid given by ______

a) 1 b) $\frac{4}{3}$ c) $\frac{5}{3}$ d) 2

11. The divergence theorem converts

a) Line to surface integral

[✓ b) surface to volume integral]

c) volume to line integral

d) surface to line integral

12. The volume of Green's theorem for $M = x^2$ and $N = y^2$ is ______

[✓ a) 0] b) 1 c) 2 d) 3

13. The stoke's Theorem used which of the following operation.

a) divergence b) Gradient [✓ c) Curl] d) Laplacian

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14. If v is the volume enclosed by the closed surface S and $\vec{F} = ax\vec{i} + by\vec{j} + cz\vec{k}$, then $\iint_S \vec{F} \cdot \hat{n}\, ds =$ ________

[✓ a) $(a+b+c)\,v$] b) $3v$ c) v d) 0

15. The area of the ellipse $x = a\cos\theta$, $y = b\sin\theta$ is ________

a) πa^2 b) π [✓ c) πab] d) $\pi(a^2+b^2)$

16. The area bounded by a principle closed curve C is the xy plane is ________

a) $\int_C x\,dx$ b) $-\int_C y\,dx$ [✓ c) $\frac{1}{2}\int x\,dy - y\,dx$] d) All the above

17. $\iiint_V \nabla \cdot (x^2\vec{i} + y^2\vec{j} + z^2\vec{k})\, dv$ if v is the volume of the region bounded the cube $0 \le x, y, z \le 1$

a) 2 b) 1 [✓ c) 3] d) 0

18. $\iint_V \text{div}\, \vec{A} \cdot d\vec{v} = \iint_S \vec{A} \cdot d\vec{s}$ is ________ theorem

a) Green's [✓ b) Gauss divergence] c) Stoke's

d) none

19) $\oint_C M\,dx + N\,dy$ = ________

a) $\iint_R \left(\frac{\partial N}{\partial x} + \frac{\partial M}{\partial y}\right) dx\,dy$

[✓ b)] $\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$

c) $\iint_R \left(\frac{\partial N}{\partial r} + \frac{\partial M}{\partial \theta}\right) dr\,d\theta$ d) None

20. $\iint_C (\text{curl } \vec{A})\ \vec{ds}$ = ________

[✓ a)] $\oint \vec{A}\,\vec{dr}$ b) $\oint_C \vec{A}\cdot\vec{dr}$ c) $\int_C \vec{A}\cdot\vec{ds}$ d) none